Queen, Clifford, *Lehigh University.*
Rhoades, B.E., *Indiana University at Bloomington.*
Ridge, Jack, *Moravian College.*
Rigby, John F., *University College, Cardiff.*
Roberts, A. Wayne, *Macalester College.*
Robertson, Timothy J., *University of Iowa.*
Rosen, David, *Swarthmore College.*
Rusin, David, *University of Chicago.*
Sallee, G.T., *University of California, Davis.*
Savage, Thomas, *St. Olaf College.*
Sawka, John, *University of Santa Clara.*
Senechal, Marjorie, *Smith College.*
Shephard, Geoffrey C., *University of East Anglia, Norwich.*
Siegel, Martha J., *Towson State Univ.*
Smogor, Louis E., *DePauw University.*
Snyder, Drew, *Lehigh University.*
Spencer, Joel, *SUNY at Stony Brook.*
Stapleton, James H., *Michigan State University.*
Steen, Lynn A., *St. Olaf College.*
Stephens, Stanley L., *Anderson College.*
Straffin, Philip D., Jr., *Beloit College.*
Thompson, Anthony C., *Dalhousie Univ.*
Thompson, Maynard, *Indiana University, Bloomington.*
Tucker, Alan, *SUNY at Stony Brook.*
Wilansky, Albert, *Lehigh University.*
Wolfe, Dorothy, *Widener College.*
Wright, Tim, *University of Missouri, Rolla-Rolla.*

AUTHORS

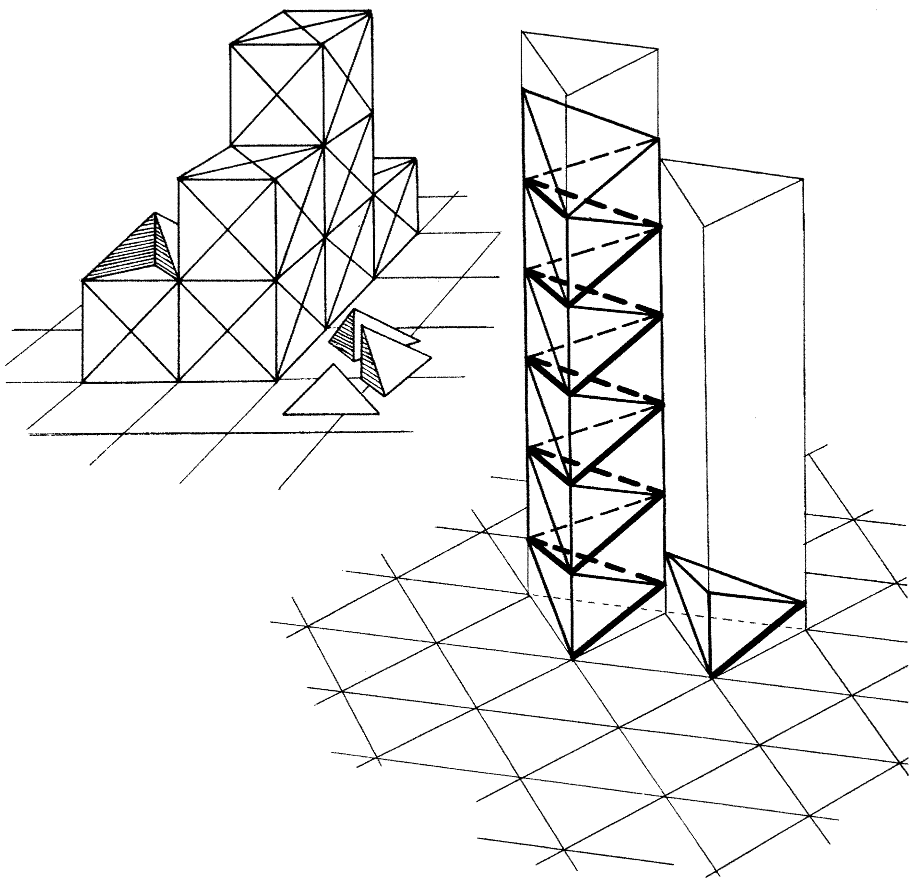
TITLES

PROBLEMS

Proposals, Solutions, and Quickies are indexed below by means of the code letters P, S, *and* Q, *respectively. Page numbers are given in parentheses. Thus,* P1129(211) *refers to proposal number* 1129 *which appears on page* 211.

NEWS

MATHEMATICS MAGAZINE



Vol. 54, No. 5
November, 1981

SPACE-FILLING TETRAHEDRA • MAKING CONNECTIONS
ELECTION REVERSAL • CHORDS IN A SQUARE

MATHEMATICS MAGAZINE

Volume 54, Number 5, November 1981
(ISSN—0025-570X)

COVER: An obvious and an unexpected packing of space by congruent tetrahedra. See **Illustrations,** p. 226; also p. 236 and p. 242. Design by the Editor.

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS

INDEX TO VOLUME 54

EDITORIAL POLICY

Mathematics Magazine is a journal of collegiate mathematics which aims to provide inviting, informal mathematical exposition of interest to undergraduate students. Manuscripts accepted for publication in the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles submitted for publication should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

New manuscripts should be sent to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, Pennsylvania 18018. Manuscripts should be prepared in a style consistent with the format of *Mathematics Magazine*. They should be typewritten and double spaced on $8\frac{1}{2}$ by 11 paper. Authors should submit the original and one copy and keep one copy as protection against possible loss. Illustrations should be carefully prepared on separate sheets of paper in black ink, the original without lettering and two copies with lettering added.

Authors planning to submit manuscripts should read the full statement of editorial policy which appears in the News and Letters section of this *Magazine*, Vol. 54, pp. 44–45. Additional copies of the policy are available from the Editor.

AUTHORS

Marjorie Senechal ("Which Tetrahedra Fill Space?") grew up in Lexington, Kentucky. She received her B.S. degree from the University of Chicago in 1960 and her Ph.D. from the Illinois Institute of Technology in 1965. The following year she began teaching at Smith College, where she is now Professor of Mathematics and chair of the department. At Smith she met the crystallographer Dorothy Wrinch who helped to focus her interests in number theory, algebra, and geometry on the theory of regular spatial arrangements and its applications. Among other things this has led to an unusual interdisciplinary course on symmetry, the book *Patterns of Symmetry* (co-edited with the chemist George Fleck) and extended research visits at the crystallographic institutes of the University of Groningen in The Netherlands and the Academy of Sciences of the USSR. She is a member of the Five-College Committee on Applied Mathematics and the Smith College Committee on the History of Science.

ILLUSTRATIONS

Janet L. Stevenson, a student at Widener College, provided the illustration of the solution to the pen-pal problem (p. 255).

The editor assisted with the illustrations for "Which Tetrahedra Fill Space?" The cover shows how M. Goldberg packs an equilateral triangular prism with congruent copies of a single tetrahedron, whose spiral path is traced by a single (heavy) line. The cube packing is obvious.

All other illustrations were provided by the authors.

BUSINESS INFORMATION. Mathematics Magazine is published by the Mathematical Association of America at Washington, D.C., five times a year in January, March, May, September, and November. The annual subscription price for Mathematics Magazine to an individual member of the Association is $8, included as part of the annual dues of $30. Students receive a 50% discount. Bulk subscriptions (5 or more copies to a single address) are available to colleges and universities for distribution to undergraduate students at a 35% discount. The library subscription price is $18.

Subscription correspondence and notice of change of address should be sent to A. B. Willcox, Executive Director, Mathematical Association of America, 1529 Eighteenth Street, N. W., Washington, D.C. 20036. Back issues may be purchased, when in print, from P. and H. Bliss Co., Middletown, Connecticut 06457.

Advertising correspondence should be addressed to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, New York 14214.

Second class postage paid at Washington, D.C., and additional mailing offices.

Which Tetrahedra Fill Space?

Early mathematicians gave some puzzling answers; today the problem is not yet completely solved.

MARJORIE SENECHAL
Smith College
Northampton, MA 01063

Filling space by fitting congruent polyhedra together without any gaps is one of the oldest and most difficult of geometric problems, and has a fascinating history. It arose first in ancient times in relation to Plato's theory of matter; during the subsequent 2300 years of its development, it has continued to receive its principal stimulus from physicists and others interested in the structure of the solid state. In its most intuitive form, the problem is that of determining the shapes of building blocks—the building blocks of architecture, of inorganic and organic matter, of space itself. Its origin can be traced to Plato's atomic theory: the hypothesis that all matter is the result of combinations and permutations of a few basic polyhedral units. The mathematical question is: *what shape must such a unit have if it is possible to fill space without gaps by figures congruent to that single unit*? This simply stated geometric problem is still unsolved, despite considerable efforts devoted to it over the ages.

That rectangular solids or, more generally, parallelopipeds can be fitted together to fill space was known to the earliest bricklayers, but that any other polyhedra have this property is less obvious. Plato, as we shall see, assumed the existence of such polyhedra, but Aristotle was the first to get down to details. In the process he made a mistake that generated a controversy lasting nearly 2,000 years.

Aristotle asserted that, of the five regular solids (FIGURE 1), not only the cube but also the tetrahedron fills space. That this is incorrect (FIGURE 2) does not seem to have been evident at that time, and many of the later Aristotelian scholars—if they realized that something was amiss—apparently assumed that somehow *they* must be mistaken. In trying to justify Aristotle's erroneous assertion, they raised the interesting question of which tetrahedra actually do fill space, and they developed some of the techniques used today in the study of space-filling polyhedra.

The early history of the space-filling problem was discussed in detail by Dirk Struik in 1925 [**1**]; there he showed how Aristotle's error, for all the confusion it caused, indirectly played a constructive role in the development of the theory of polyhedral angles. The story is instructive in many ways. It shows how errors can arise through misunderstanding of a problem or excessive deference to a great thinker, and how they can be perpetuated for these reasons or through simple carelessness, sometimes even after the problem has been properly resolved. The first section of this paper is based on Struik's article. In the second, we briefly sketch the important role that the problem has played in the development of the theory of the structure of crystals from about 1600 to the present. In this discussion we hope to show that the interaction between geometry and natural sciences can be profitable for both sides. Finally, we discuss the question inadvertently raised by Aristotle: *which tetrahedra fill space and which do not*?

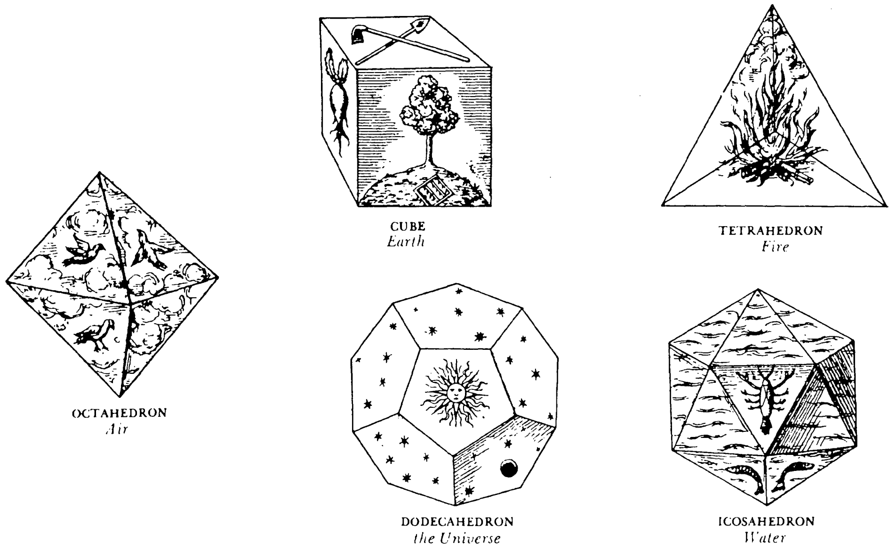


FIGURE 1. **The five regular solids as depicted by Johannes Kepler in *Harmonices Mundi, Book II* (1619). Redrawn by John Kyrk, this is Illustration 1 in [3]. (Reprinted with permission.)**

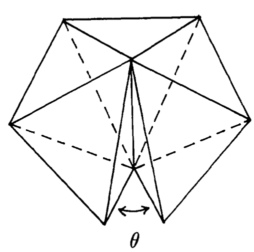
§1. An error for almost 2000 years

The discovery of the regular solids, and the proof that there are exactly five of them, was one of the great mathematical achievements of the ancient Greeks. They were discussed in detail by Euclid in the final book (XIII) of the *Elements*; it has even been suggested that the purpose of the *Elements* was to provide a rigorous treatment of their construction. Plato seems to have been the first to "apply" the theory of these polyhedra in the interpretation of nature: they were the basis of his theory of matter, which is presented in his dialogue *Timaeus*. According to Plato, all matter consists of combinations of four basic "elements": earth, air, fire, and water (this corresponds rather well to our present concept of the phases of matter). The elements of each type are composed of particles, "far too small to be visible," of definite shape: the earth particles are cubes, the water particles regular icosahedra, the air particles regular octahedra, and those of fire, regular tetrahedra. (The fifth regular solid, the pentagonal dodecahedron, was associated with the cosmos.) The varieties of the elements (such as different kinds of stone, or different liquids) were explained on the hypothesis that the basic particles come in many different sizes, while substances which are mixtures of elements were assumed to consist of mixtures of the corresponding particles.

Aristotle argued that Plato's theory is incompatible with reality. If a substance is composed of particles of a given shape and size, then these particles must pack together to fill the space

FIGURE 2.

The regular tetrahedron does not fill space without gaps. Its four faces are equilateral triangles, from which it follows that its dihedral angles α (the angles between adjacent faces) are equal to arccos $(1/3)$, or $\alpha \approx 70° 32'$. If 5 tetrahedra are fitted around an edge, there is a gap whose angular measure θ is less than α, and we conclude that regular tetrahedra do not fill space when arranged face-to-face. In any other arrangement a dihedral angle of $\pi - \alpha$ is created, which cannot be filled by regular tetrahedra.



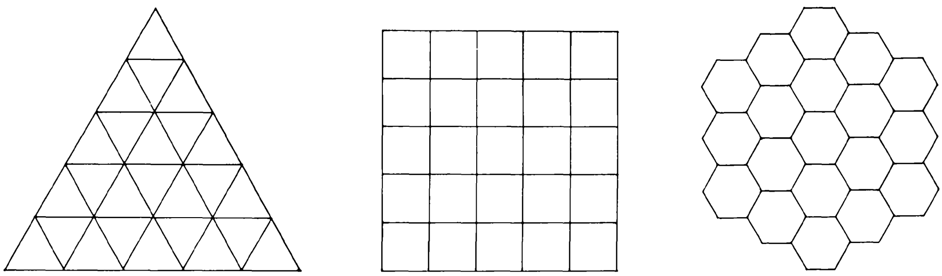


FIGURE 3. **The plane can be filled with squares, equilateral triangles, and regular hexagons. No other regular polygons can fill the plane without gaps.**

occupied by the substance, that is, they must fill space without leaving any gaps. A gap would mean empty space, or a vacuum, which according to the Aristotelian theory of motion cannot occur in nature. But some of the regular solids do not fill space. Thus, remarked Aristotle, "In general it is incorrect to give a form to each of the singular bodies, in the first place, because they will not succeed in filling the whole. It is agreed that there exist only three plane figures that can fill a place, the triangle, the quadrilateral, and the hexagon, and only two solid bodies, the pyramid and the cube. But the theory demands more than these, because the elements they represent are greater in number" (quoted from *De Caelo* III, 306b). This was considered to be a serious argument against the ancient atomic theory, which consequently became increasingly unpopular.

We may assume that Aristotle was referring to the fact that the only *regular* polygons which fill the plane with copies of themselves are the square, the equilateral triangle, and the regular hexagon (FIGURE 3). From this and from the context of his remark, we conclude that Aristotle believed that the regular octahedron and icosahedron do not fill space (in this he was correct) while the cube and regular tetrahedron do. He gave no evidence for his claim. Struik remarks, "This passage, which is only reported incidentally in a modern investigation of Aristotle's mathematics, caused the ancient writers considerable concern." Thus we find a series of commentators on Aristotle discussing the number of tetrahedra that can "fill the space about a point," that is, be packed together so as to share a vertex. Simplicius, a scholar and commentator who lived in the first half of the sixth century A.D., asserted that the number of such tetrahedra is twelve, but gave no reason. He also stated that Potaman (a philosopher who probably lived in the first century A.D.) had concluded that the number was eight, by the following reasoning. The maximum number of cubes which can share a vertex is eight. If we truncate each of the cubes meeting at that vertex, we obtain eight tetrahedra which fill space about a point (FIGURE 4).

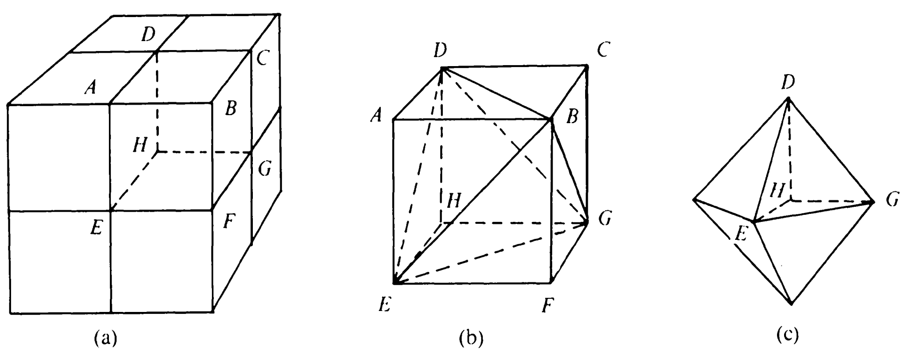


FIGURE 4. **(a) Cubes can pack space, eight meeting at a single vertex. (b) Each cube can be partitioned into five tetrahedra: *BGCD, EFBG, ABED, DEHG,* and *DBEG*. This is probably the construction Potaman had in mind. Only the "central" tetrahedron *DBEG* is regular; the other four are congruent "corners" of the cube. (c) Eight tetrahedra congruent to *DEHG* will pack with all right angles meeting at a single point to form a regular octahedron.**

Potaman's argument, as reported here, has some defects which illustrate the sort of difficulties which have plagued this problem throughout its history. In the first place, the eight tetrahedra Potaman obtains by truncating eight packed cubes are not regular: the tetrahedral faces which meet at the vertices of the cube are right, not equilateral, triangles. While Aristotle did not state explicitly that he meant regular tetrahedra, it is, as we have seen, reasonable to assume that this is what he intended. Even if we ignore this objection, another remains: filling the space about a point is *not* the same thing as filling space as a whole. Some packing arrangements cannot be continued to fill all of space. It is possible that Potaman thought that if his "truncation" procedure was carried out on all the cubes in a regular packing of cubes, then each cube would be partitioned into congruent tetrahedra; if this were so, then the tetrahedra would be space-fillers. While his construction does dissect each cube into five tetrahedra, these five tetrahedra are not congruent: the "central" tetrahedron is regular. Also an argument similar to that in the caption to FIGURE 2 shows that his "vertex tetrahedra" do not fill all space. Nevertheless, Potaman's technique, partitioning a known space-filler into congruent parts, is one of the most useful we have for constructing new space-filling polyhedra.

Aristotle's error not only stimulated the development of this technique; it also led to the earliest attempts to define the measure of a polyhedral angle. The 12th century Arabic commentator on Aristotle, Averroës, seeking to justify Aristotle's remark, developed a theory of angle measure which beautifully served its purpose. According to Averroës, the measure of a trihedral angle (such as the angle at the vertex of a cube or of a tetrahedron) is the sum of the face angles that form it. Thus the measure of a vertex angle of a cube is 270°, because each of the three face angles has measure 90°, and the measure of a vertex angle of a regular tetrahedron is 180°, because three 60° angles sum to 180°. Now, he reasoned, since eight cubes fill the space about a point, a necessary and sufficient condition for space-filling by tetrahedra is that the sum of the trihedral angles meeting at a point be equal to the product $8 \times 270°$. Since $12 \times 180 = 8 \times 270$, it follows that twelve regular tetrahedra fill the space about a point, in agreement with Simplicius.

Averroës's theory of angle measure was generalized by the 13th century English Franciscan scholar Roger Bacon to include polyhedral angles formed by four or more planes. In the course of this, Bacon discovered that Aristotle had missed something: the measure of a vertex angle of an octahedron is $4 \times 60° = 240°$, and $9 \times 240 = 8 \times 270$, whence nine octahedra fill space about a point! Bacon regarded this result as a significant advance. But he was aware that there was some controversy about this question since in Paris "a fool had asserted in public" that twenty pyramids fill the space about a point. Bacon added that to settle the matter it would be necessary to understand Euclid's Book XIII. But, Struik says, "It is indicative of what men in that time could and could not do, that they preferred lengthy disputes to either calculating according to Euclid or taking the trouble to construct a single model."

Actually it is intuitively reasonable that the measure of a polyhedral angle should be, in some way, closely related to the sum of the face angles comprising it, but the relationship is not as simple as these scholars supposed (see below). That something was wrong with the Averroës-Bacon theory was first pointed out by the English scholastic Thomas Bradwardinus (1295-1349): if twelve regular tetrahedra filled the space about a point, then they would together form a convex polyhedron with twelve equilateral triangular faces, which would be a sixth regular solid. Also, he noted, the theory must be incorrect because Aristotle did not include the octahedron in his list of space-fillers. According to Bradwardinus, Aristotle's tetrahedra could be obtained by joining the vertices of a regular icosahedron to its center; in this way we obtain the twenty tetrahedra of the Parisian fool. Bradwardinus was unsure whether these tetrahedra were regular. (They are not, but unlike Potaman's tetrahedra this is not obvious: detailed calculations are needed to establish the fact that the ratio of the length of the icosahedral edge to the vertex-center distance is approximately 1.05.) Perhaps Bradwardinus, like some later commentators, would have accepted the nonregularity of these tetrahedra on the grounds that Aristotle did not explicitly require regularity. He appears not to have inquired whether this packing arrangement could be repeated to fill all of space (it cannot, since the icosahedron is not a space-filler).

Bradwardinus's refutation of the Averroës-Bacon theory was a significant achievement, but it should be pointed out that it is not strictly correct. The regular solids are characterized by the additional requirement that the same number of faces meet at each vertex. If two regular tetrahedra are juxtaposed along a face, we obtain a polyhedron whose six faces are equilateral triangles, but it is not a regular solid. There are four other "irregular" convex polyhedra (sometimes called "deltahedra" [3]) all of whose faces are equilateral triangles, including one with twelve faces.

Bradwardinus's argument could have been extended to prove more rigorously that nine octahedra do not fill the space about a point. For if we cut the octahedra by planes passing through the endpoints of the edges meeting at the point, we obtain a convex polyhedron with nine square faces, which is impossible. This suggests another way of deciding whether or not a given arrangement of polyhedra fills the space about a point: find the shortest polyhedral edge e which meets the point, then draw a sphere about the point, choosing the radius r of the sphere to satisfy $r \leq e$. In this way we obtain a partition of the sphere into spherical polygonal regions whose edges are the traces on the sphere of the polyhedral faces which meet at the point, and whose vertices are the points at which the sphere cuts the edges which bound them. Later, when spherical trigonometry had been developed, the measure of a polyhedral angle was correctly defined to be the area of the spherical polygon found in this way. Thus a necessary and sufficient condition for the polyhedra to fill the space about a point is that the sum of the areas of these polygons equal the surface area of the sphere.

Only in the 15th century, when Euclid was again studied, did the confusion begin to be resolved. Johannes Müller, or Regiomontanus (1436–1476), the author of an important work on spherical trigonometry, was the first to discuss the problem in a critical spirit, as we can tell from the lengthy title of his manuscript, "On the five like-sided bodies, that are usually called regular, and which of them fill their natural place, and which do not, in contradiction to the commentator on Aristotle, Averroës." Unfortunately this work was lost, but subsequent authors, probably influenced by him, discussed the problem in a similar way, pointing out that it is clear from Euclid's construction of the icosahedron that the tetrahedra obtained from it are not regular. They also noted that together six regular octahedra and eight regular tetrahedra fill the space about a point (this is implicit in Potaman's construction: the bases of the eight tetrahedra meeting at a cube vertex together form a regular octahedron; see FIGURE 4). Except for the cube and combinations of the tetrahedra-octahedra packing with the cube, there are no other ways to fill space with regular solids.

Despite this criticism, Aristotle's error in its various guises persisted for a long time afterwards; scientists who should have known better perpetrated it by carelessly accepting the earlier fallacious arguments. Even when the error was finally generally admitted, some scholars continued to defend Aristotle on the grounds that he had not explicitly required regularity. The correct formula for the area of a spherical polygon was first published in a book by Albert Girard, in 1629. Later, the Polish mathematician J. Broscius (1591-1652) devoted a large portion of an important book to a thorough discussion of this question. In the course of his argument he developed a formula for the area of a spherical polygon and it is for this that the book is best known today. Here at last the problem of filling space about a point was discussed correctly and in detail.

This achievement came at the time when science was turning from speculation to experiment. The structure of matter again became a focus of interest. In the study of crystals, the problem of filling space with congruent polyhedra took on a new significance.

§2. Crystallographers revive the problem

Plato had not been concerned with the problem of how *external form* is achieved by the stacking of particles. It is in the sixteenth and seventeenth centuries that we find the first investigations of this issue. The great astronomer Johannes Kepler (1571-1630), who also made an important contribution to the problem of filling the plane with polygons, became interested in the

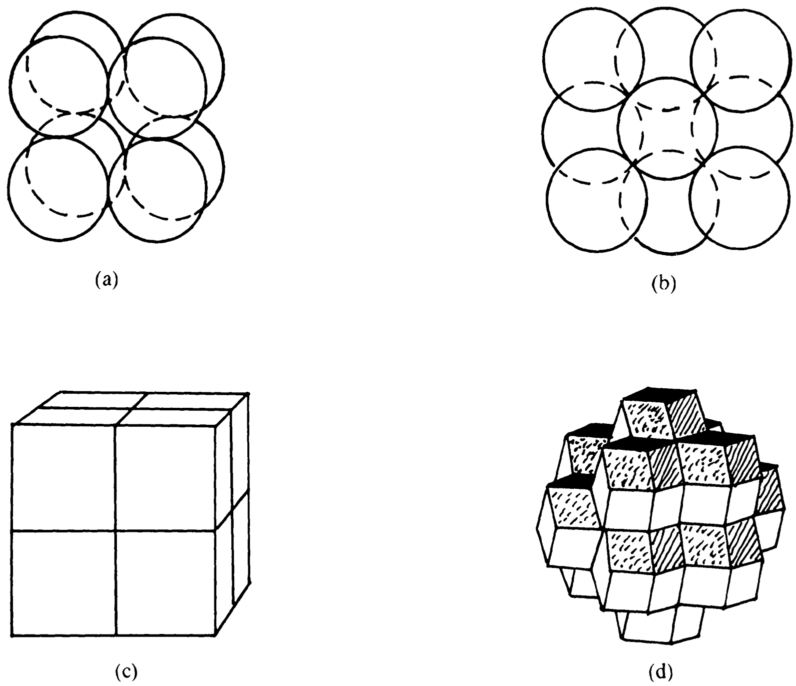


FIGURE 5. **(a) In simple cubic packing, spheres are arranged at the vertices of cubes; each sphere touches six others. (b) In face-centered cubic packing, spheres are arranged at the vertices and face centers of cubes; each sphere touches twelve others. (c) When packing (a) is compressed, the spheres are deformed into polyhedra with six faces (cubes). (d) When packing (b) is compressed, the spheres are deformed into polyhedra with twelve faces (rhombic dodecahedra).**

causes of the forms of snowflakes and wrote a booklet about his ideas as a New Year's gift to a friend in 1611 [9]. Postulating that snowflakes are composed of minute spheres of ice, he studied various packing arrangements for spheres and also some of the polyhedral forms that spheres would assume if the packing arrangements were uniformly compressed. In this way, he discovered several space-filling polyhedra. For example, if the spheres are arranged in what is known as simple cubic packing, the compression forms are cubes; if they are arranged in the so-called face-centered cubic packing, the compression forms are rhombic dodecahedra (FIGURE 5). It is important to note that in Kepler's work on snowflakes we find a completely new approach to the space-filling problem. All of the authors whose work was described in the preceding section were principally concerned with fitting given polyhedra together locally—specifically, matching faces at a vertex. (Kepler also took this approach, in his study of plane tilings.) The properties of the spatial patterns which could be generated by extending these local packings do not seem to have been considered important (indeed, as we have seen, some authors did not even investigate whether an extended pattern existed). The compression polyhedra Kepler obtained from sphere-packings, however, were solutions of a global problem: what kinds of polyhedra pack together according to the requirements of a given repeating pattern?

By this time the atomic hypothesis had been revived and was being vigorously debated. Sphere-packing was a popular approach to the study of matter. In 1665 the English scientist Robert Hooke stated that he could show that all crystalline forms could be explained by a few basic packing arrangements of spherical atoms ("had I time and opportunity") and gave several examples. But spheres, even when packed together as closely as possible, still leave gaps; the vacuum problem was a persistent difficulty (although the existence of a vacuum had been demonstrated in 1643). One way to get around it was to assume that atoms are not spherical but polyhedral in form. The first post-Platonic theory of this sort was that of the Italian physician and mathematician Domenico Guglielmini (1655–1710) who was interested in the structure of salts. There are, he said, four principal types of salts, and the atoms of each have the form of a polyhedron: a cube, a hexagonal prism, a rhombohedron, or an octahedron. The basic salts are constructed of atoms of a single shape; other salts are formed by combinations of these atoms. We see here that the theory of polyhedral atoms, which was attractive for many reasons, did not solve the problem it was intended to solve, since (regular) octahedra do not fill space—as Guglielmini was aware.

Both the sphere-packing and polyhedra-packing theories were based on the assumption that the external geometry of crystals is the result of some sort of structural regularity, despite the fact that the forms of the crystals of a given mineral species can vary greatly (FIGURE 6). For many years this view was as controversial as the atomic hypothesis on which it was based. In about 1782, however, it was discovered empirically that there is a definite relationship between the various polyhedral forms assumed by the crystals of a species. This strongly suggests that the external form of crystals is a reflection of something fundamental and characteristic.

This relationship is known to crystallographers as the Law of Constancy of Interfacial Angles; it was stated in its most general form in 1783 by the French mineralogist J. B. L. Romé de l'Isle (1736–1790). It can be described in the following way. The faces of a polyhedron can be grouped together in families which are related by symmetry. As we saw in FIGURE 6, the faces of a given family may be larger or smaller in one individual crystal than in another, or even entirely absent, but in every case the interfacial angles are exactly the same, and are characteristic of the species. Romé pointed out that for each crystal species, the families of faces can be obtained from a single basic form by truncating its vertices and edges, and devoted a book to demonstrating this. (Truncation still seems to hold a tremendous fascination for many people.)

Romé himself believed that speculation about the structure of crystals was premature and so the study of the implications of his work was left to others. Not long afterward the French abbé and mineralogist Rene Just Haüy (1743–1827) revived and expanded the polyhedral theory of crystal structure; his work marks the beginning of the modern science of crystallography.

We will note here only the chief geometrical features of Haüy's theory. The building blocks of crystals are polyhedra which can be regarded as "crystal molecules" (which are not necessarily the same units as chemical molecules). These polyhedra may be parallelopipeds, tetrahedra, or triangular prisms: the type is characteristic for a crystal species. The "nucleus" of a crystal consists of several of these polyhedra grouped together, and the crystal grows by the accretion of

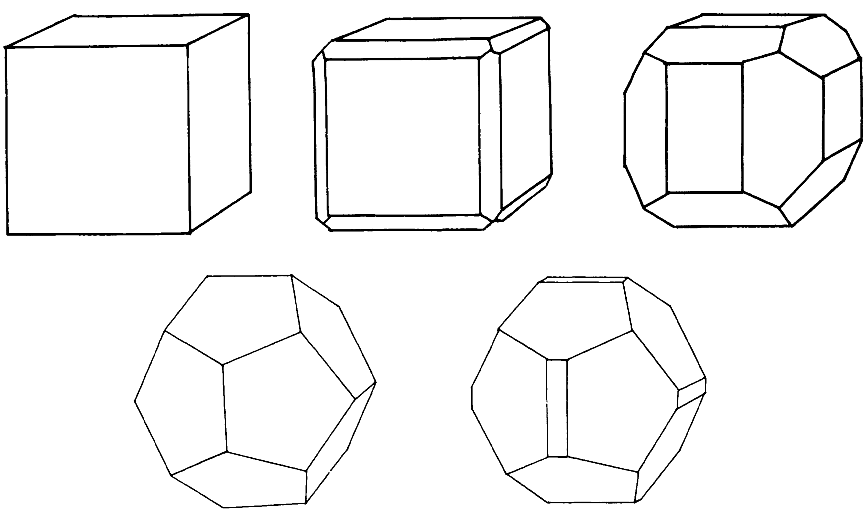


FIGURE 6. **Adapted from Goldschmidt's *Atlas der Kristallformen*. Five crystals of pyrite. The dodecahedron at lower left is not regular.**

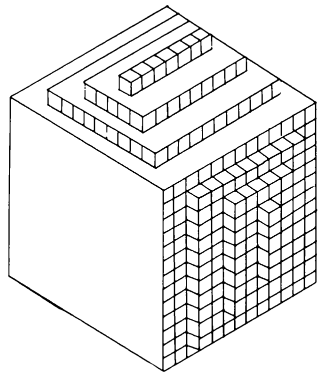


FIGURE 7. **Haüy's construction of pyrite from parallelopiped blocks.**

more of them, now grouped together to form parallelopipeds (FIGURE 7). Some crystal faces are thus smooth and others stepped, but the latter appear smooth to us because of the submicroscopic size of the steps. On this hypothesis Haüy was able to explain the forms that crystals assume, and why other forms (such as the regular icosahedron and pentagonal dodecahedron) cannot exist in the crystal world. He was also able to give an explanation of certain physical properties of crystals, such as cleavage.

Nevertheless, Haüy's theory was controversial. His constructions did not always agree very well with measurement, which provoked severe criticism of his work. The arguments centered on the reality of his building blocks. Only later was their value as an abstract model understood. Haüy's work led to the modern concept that periodicity—regular repetition in all directions—is the fundamental structural characteristic of crystals.

The first steps toward this abstraction were taken by the German physicist Ludwig Seeber in 1824. Pointing out that Haüy's theory could not explain the expansion and contraction of crystals with changes of temperature, he proposed replacing the parallelopiped blocks by a system of points representing their centers of gravity, which he called the **space lattice**. This model has been exceptionally fruitful.

Lattices can differ in their symmetry and in the way in which the points are arranged. (In 1849 August Bravais proved that there are exactly fourteen types.) The points of a space lattice can be represented by the endpoints of vectors of the form $\vec{v} = \vec{a}x + \vec{b}y + \vec{c}z$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three linearly independent vectors and x, y, z range over the integers. If we calculate $|\vec{v}|^2$ we obtain a homogeneous quadratic form in three variables. Conversely, each such form represents a space lattice. But different forms can represent the same lattice: how can we tell which forms are equivalent? This question is closely related to the number-theoretic problem of the "reduction" of quadratic forms, that is, the identification of the forms by the parameters $\vec{a}, \vec{b}, \vec{c}$. It is of interest to us because a major advance in the space-filling problem came indirectly from the reduction problem, of which Seeber himself was the first to find a solution (1830). Seeber's work was correct but aesthetically unappealing: it appeared to be unnecessarily long and complicated. This prompted efforts by several mathematicians to simplify it, and it was in the course of this that P. Dirichlet introduced a construction for the region of space closer to a given lattice point than to any other (FIGURE 8). If the **Dirichlet region** is constructed for each point of a space lattice, we obtain a filling of space by congruent convex polyhedra whose centers lie at lattice points.

The polyhedra which form Dirichlet regions of a space lattice are related to one another by translation. In general, polyhedra which lead to space fillings by translation are called **parallelohedra**. Parallelohedra are the building blocks of space lattices and thus are clearly important for theoretical crystallography. But for many years there were controversies about their physical interpretation. Is crystal structure really periodic? If so, is the block structure really an appropriate model? How are the blocks related to chemical structure? And so forth.

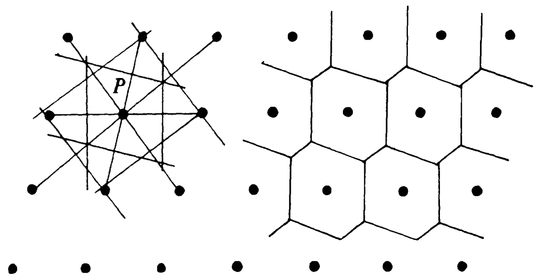


FIGURE 8. **Given a discrete point set, the Dirichlet region of a given point *P* is the region of space closer to *P* than to any other point of the set. It can be constructed by joining *P* to each of the other points by straight line segments, bisecting the segments, and finding the smallest convex region bounded by the bisectors. The Dirichlet regions of a space lattice are congruent polyhedra in parallel orientation which fit together face-to-face to fill space without gaps. (Here the Dirichlet regions of some of the points of a plane lattice are shown.)**

The great Russian crystallographer and geometer E. S. Fedorov (1853–1919) believed that the parallelohedra into which a crystal can be partitioned contain groupings of chemical molecules, and that by partitioning the parallelohedra in turn into congruent regions (Potaman's principle) we obtain the true subunits of the crystal. Fedorov investigated parallelohedra in detail, and proved the remarkable theorem that the convex parallelohedra can be classified into five topological types: the cube, the hexagonal prism, the rhombic dodecahedron, a dodecahedron with eight rhombic and four hexagonal faces, and the truncated octahedron (FIGURE 9). This was the first general result in the theory of space-filling polyhedra and is still the most important one. Fedorov included this theorem, along with many other interesting results, in a book, *An Introductory Study of Figures*, which he had begun at the age of 16. It was finally published in 1885, after being turned down by several mathematicians. (P. Chebyshev refused even to read it, remarking that "contemporary science is not interested in these questions.") Fedorov's book has never been translated into a western language, but a simplified proof of his theorem is contained in [7].

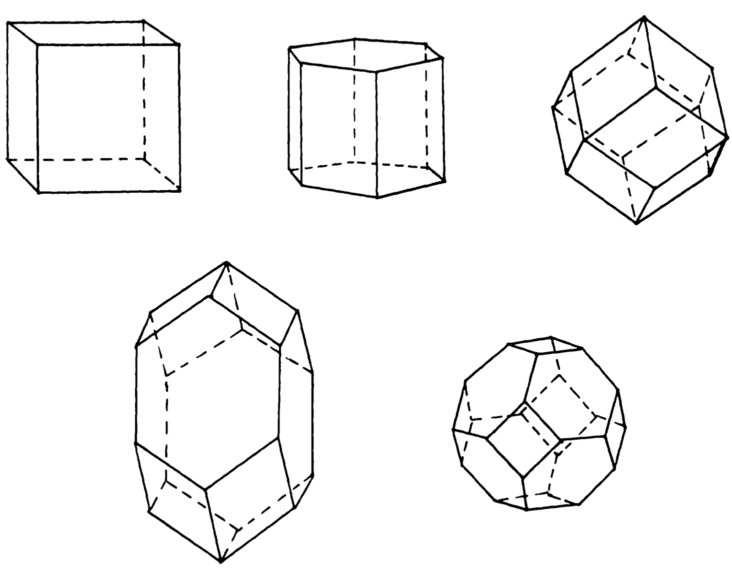


FIGURE 9. **Fedorov's five topological types of parallelohedra.**

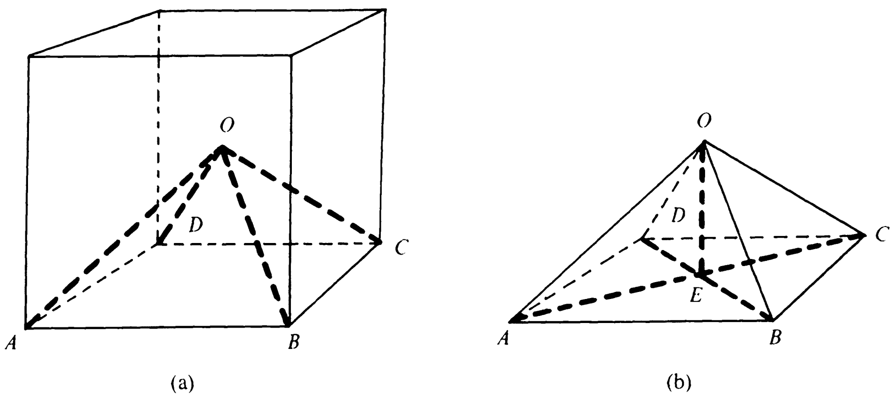


FIGURE 10. **The cube can be partitioned into twenty-four congruent tetrahedra. Let E be the center of the face $ABCD$ and O the center of the cube. (a) Joining A, B, C, and D to O forms a pyramid. (b) Joining A, B, C, D, and O to E partitions the pyramid into four congruent tetrahedra. Repeating this construction for each of the cube faces, we obtain the remaining twenty congruent tetrahedra.**

In 1912 the space-lattice theory of crystals was validated by x-ray techniques. Although Fedorov's view of the molecular groupings in crystals turned out to be incorrect, the space-filling model continues to provide useful interpretations of crystal structures. Thus the space-filling problem is a subject of active research by crystallographers as well as by mathematicians. Dirichlet regions are of particular interest; with the aid of computers they can now be constructed for complicated sets of points. Some of these regions are tetrahedra, and so provide some answers to our title question.

§3. Which tetrahedra fill space?

We are back to our title question, and in this section examine what answers are known. The techniques used to search for answers have their origins in the history we have outlined in the previous sections. Let us begin with an example reminiscent of Potaman's technique: a dissection of a cube—but into congruent tetrahedra. If we join each vertex of a cube to its center, it is dissected into six congruent pyramids. Each pyramid can be further dissected into four congruent tetrahedra by joining each of its vertices to the center of its square face. Thus we obtain a partition of the cube into twenty-four congruent tetrahedra (FIGURE 10). Other space-filling tetrahedra can be found by further partitioning this one (FIGURE 11). In the search for space-filling polyhedra, it seems logical to begin searching for such tetrahedra, since the smallest number of faces are involved. But even today, this problem is not solved, nor has a comprehensive technique to discover and enumerate all such tetrahedra been developed.

It is important to note that there is no a priori reason why space-filling tetrahedra must satisfy either the local requirements imposed by the authors discussed in Section 1 or the global requirements of those discussed in Section 2. We shall see that there are space-filling tetrahedra which do not pack together along whole faces; it is possible that there are even space-filling arrangements in which the tetrahedra are not grouped together into the units of a repeating pattern. However, the imposition of local or global requirements (or both) makes the space-filling problem mathematically tractable (to some extent). Potaman's principle, Bradwardinus's technique, and Dirichlet's construction are still the principal tools available at the present time.

Our discussion will be simplified if we first consider the various ways in which a tetrahedron can be partitioned into smaller ones. A geometric figure is said to be symmetrical if there is at least one rigid motion (or symmetry operation) that can be performed on it that leaves the appearance (and apparent position) of the figure unchanged. (For example, in the plane, if a parallelogram is rotated 180° about its center, in its new position the figure appears exactly as it did in its original position. Only if a distinguishing mark such as a label on a vertex were added

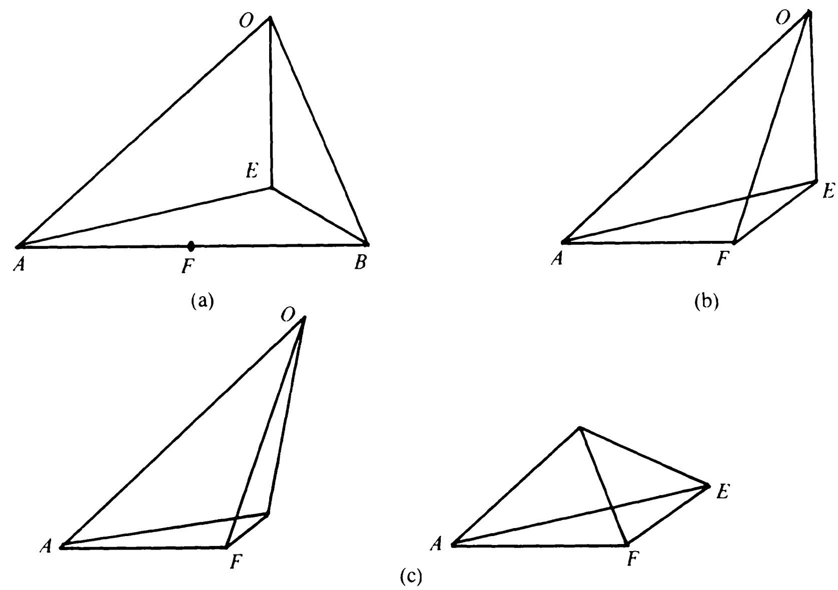
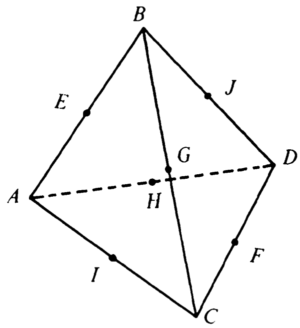


FIGURE 11. (a) The tetrahedron of FIGURE 10, with F the midpoint of edge AB. If we take the edge length AB to be equal to 2, then $AE = BE = \sqrt{2}$, $OB = OA = \sqrt{3}$ and $OE = EF = AF = BF = 1$. The tetrahedron is symmetric about the plane through E, O, and F. (b) The plane through E, O, and F divides the tetrahedron into two mirror-image tetrahedra, $AEOF$ and $BEOF$ (we show only $AEOF$). Each has a two-fold rotation axis—the axis of the tetrahedron $AEOF$ passes through the midpoints of AO and EF. (c) The tetrahedron $AEOF$ can be partitioned into two congruent tetrahedra in two ways: by a plane through A, O, and the midpoint of EF, and by a plane through E, F, and the midpoint of AO.

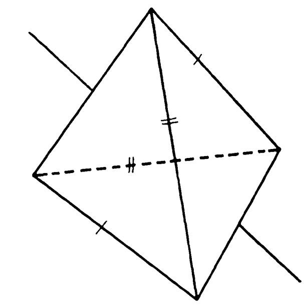
could you tell whether or not the parallelogram had been moved. Reflection in a diagonal of a parallelogram is not a symmetry operation unless the parallelogram is a rhombus.) A symmetrical object can be partitioned into congruent parts which the symmetry operation maps onto one another; if it has several symmetry operations, then there may be several ways to do this. The tetrahedron has the unusual property that these parts may be chosen to be tetrahedra. The relation between the symmetry of a tetrahedron and the ways in which it can be partitioned is shown in TABLE 1. Notice that if the operation is reflection in a plane or rotatory reflection (i.e., a rotation followed by a reflection as described in TABLE 1), then we obtain mirror-image pairs which, unless these new tetrahedra themselves have mirror symmetry, differ in the same way that right- and left-handed coordinate systems do. Since reflection cannot actually be performed in three-dimensional space, reflection is sometimes called an "improper" motion. Some authors distinguish between tetrahedra that fill space with "properly" congruent copies, and those, such as the tetrahedra of FIGURE 11(b), which must be accompanied by their mirror images.

The first systematic study of space-filling tetrahedra was carried out by D. M. Y. Sommerville (1879–1934), a geometer with deep interests in many fields of science. According to *The Dictionary of Scientific Biography*, "crystallography held a special appeal for him and crystal forms doubtless motivated his investigation of repetitive space-filling geometric patterns." The immediate inspiration for his study of tetrahedra was, however, an error made by a student. Sommerville wrote, "In the answer to the book-work question, set in a recent examination to investigate the volume of a pyramid, one candidate stated that the three tetrahedra into which a triangular prism can be divided are *congruent*, instead of only equal in volume. It was an interesting question to determine the conditions in order that the three tetrahedra should be congruent, and this led to the wider problem—to determine what tetrahedra can fill up space by repetitions" [**22**] (FIGURE 12). Sommerville wrote two papers on the subject. The first dealt with the wider problem; in the second he was concerned with the partition of triangular prisms into congruent tetrahedra, and whether these tetrahedra could fill all space.

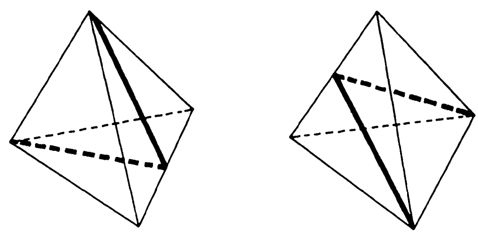
TABLE 1. **Tetrahedral symmetry and partitions of tetrahedra.**

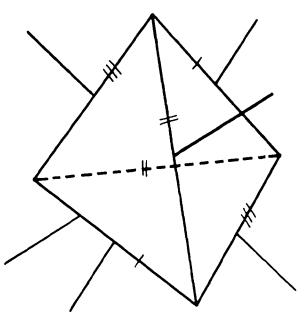


All tetrahedra described are represented schematically by the drawing at the left. Vertices of the tetrahedra are labeled as shown, with letters A, B, C, and D. Midpoints of the edges AB, CD, BC, AD, AC, and BD, are denoted E, F, G, H, I, and J, respectively. To show two edges are the same length, we mark them with the same symbol (either /, //, or ///), as is customary in elementary geometry.

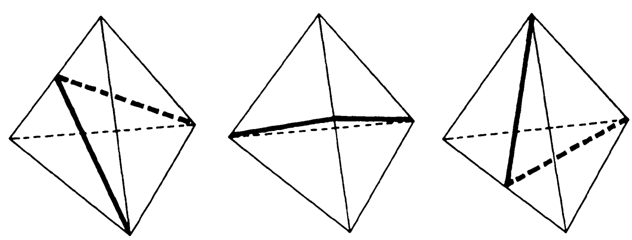


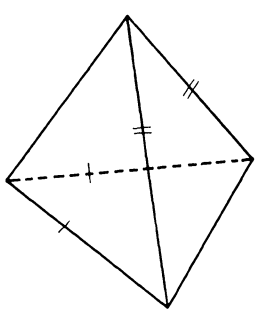
1. Symmetry 2. A single two-fold (180°) rotation axis through E and F. The tetrahedron can be partitioned into two asymmetric congruent tetrahedra in two ways, by plane ABF or plane CDE.





2. Symmetry 222. Three mutually perpendicular two-fold axes, through E and F, H and G, I and J. Each of these axes permits a partition of the tetrahedron into two congruent tetrahedra as described in 1. Since opposite edges are equal, each axis produces just one partition.





3. Symmetry *m*. A single mirror plane. This tetrahedron has two mirror-image scalene faces, ABC and BAD, and two isosceles faces, BDC and ACD. A plane through A, B, and F divides the tetrahedron into two asymmetric mirror-image tetrahedra.

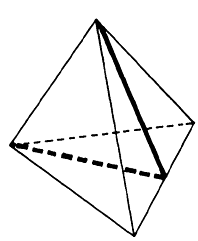
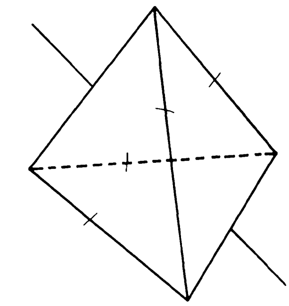
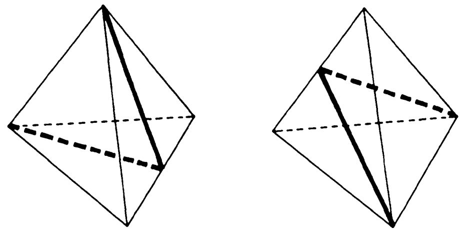
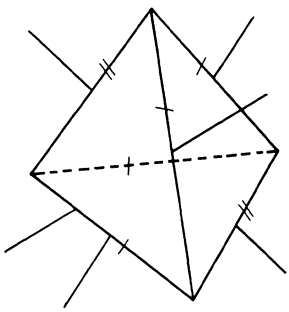


TABLE 1. **Tetrahedral symmetry and partitions of tetrahedra.**

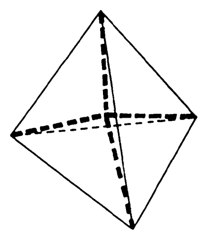


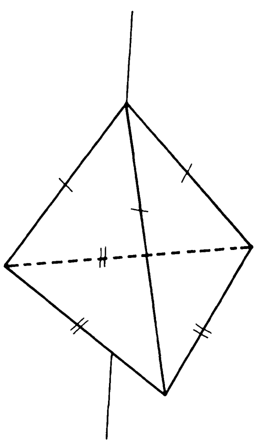
4. Symmetry $2m$. This tetrahedron has **a two-fold axis** ***EF*** **which is the intersection of two perpendicular mirror planes** *ABF* and *ECD*. It can be partitioned in two ways into congruent tetrahedra with symmetry *m*.



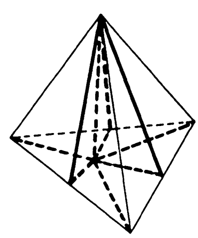


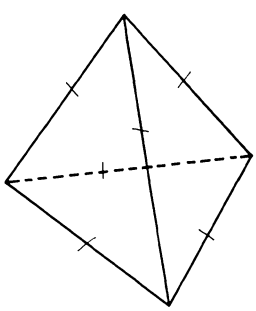
5. Symmetry $\bar{4}m$. This symmetry class contains classes 1–4: it has three two-fold axes and two mirror planes. It is generated by reflection in one of the mirror planes and by **a four-fold rotatory-reflection (denoted by $\bar{4}$): rotation 90° about** ***EF*** **followed (nonstop) by reflection in the plane through the center** ***O*** **of the tetrahedron perpendicular to** ***EF*.** This operation maps the four faces of the tetrahedron onto one another cyclicly. Thus if we join the vertices to *O* we obtain four congruent tetrahedra with symmetry *m*: *ABCO, BDCO, BADO,* and *ACDO.*





6. Symmetry $3m$. This tetrahedron has **a three-fold axis** *BQ*, where *Q* is the center of face *ACD*, and **three mirror planes passing through it.** The tetrahedron can be partitioned into six asymmetric congruent tetrahedra.





7. The regular tetrahedron. All faces are equilateral. Each of *EF*, *HG*, and *IJ* is a $\bar{4}$ axis, and there are six mirror planes. The lines from each vertex to the center of the opposite face are three-fold (120°) rotation axes. The tetrahedron can be partitioned in all of the ways shown above—and in other ways as well (the discovery of which we leave for the reader).

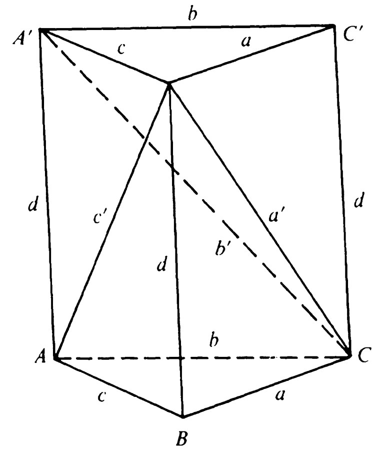


FIGURE 12. **A triangular prism $ABCA'B'C'$ can be partitioned into three tetrahedra, $ABCB'$, $B'A'CA$, and $A'B'C'C$.**

In the first paper, Sommerville defined a space-filling tetrahedron to be one which

(a) fills space with properly congruent copies such that
(b) the tetrahedra are juxtaposed face-to-face.

If a face of a tetrahedron is equilateral or isosceles, then it can be matched to the corresponding face of a properly congruent copy; otherwise it can be matched only if its mirror image also appears on the tetrahedron (consequently the tetrahedron itself must have mirror symmetry). With these definitions and observations, Sommerville classified space-filling tetrahedra into two kinds: (1) tetrahedra with mirror symmetry, and (2) tetrahedra without mirror symmetry all of whose faces are isosceles (FIGURE 13).

Sommerville then addressed the problem from both the global and local viewpoints. First, he applied Potaman's principle to the cube and discovered four tetrahedra of the first kind (FIGURE 14). Although, as we have seen, some of these tetrahedra can be partitioned further, Sommerville did not do so, evidently because the resulting tetrahedra would not satisfy condition (a). He then considered, in general, the ways in which tetrahedra of the first kind can be fitted together at a vertex. Using Bradwardinus's technique, he enumerated the triangular patterns that these tetrahedra would define on the surface of a sphere, and concluded that the four he had found the other way were the only ones possible. For some unexplained reason, however, he did not carry out a similar enumeration for tetrahedra of the second kind. Thus his claim to completeness is not justified.

FIGURE 13. **According to Sommerville, there are two mutually exclusive requirements for a tetrahedron to fill space by the juxtaposition of properly congruent copies face-to-face: (1) the tetrahedron has mirror symmetry, or (2) the tetrahedron does not have mirror symmetry but all its faces are isosceles.**

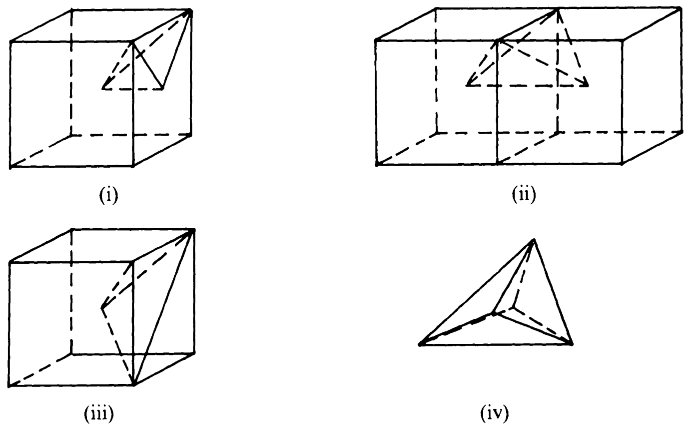


FIGURE 14. The four Sommerville space-filling tetrahedra. (i) The first tetrahedron is that of FIGURE 10. (ii) The second is found by joining two vertices of a cube which share a common edge to the centers of two adjacent cubes, as shown. (If this tetrahedron is bisected along the cube face, we obtain two tetrahedra of the first type.) (iii) The third tetrahedron is obtained from the first by joining two tetrahedra along a common face through a cube vertex, face center and cube center. (iv) Since the second has $\bar{4}m$ symmetry, we can subdivide it into four congruent tetrahedra, each of which has mirror symmetry, as described in TABLE 1.

In the second paper Sommerville showed that if the tetrahedra into which a triangular prism can be partitioned (FIGURE 12) are congruent, then $a = b' = c'$ and one of these additional relations among the edges must hold:

(i) $a' = b = c = d$, $3a^2 = 4b^2$,

(ii) $a' = b = c$

(iii) $a' = b = d$ or equivalently $a' = c = d$,

(iv) $b = c = d$.

The four families of prisms defined by these relations are shown schematically in FIGURE 15.

He then asked whether these tetrahedra could fill all of space. The tetrahedron obtained from family (i) is the second of the four that he had found in the first paper, and he showed how the other three could be derived from it by partition. In each of families (iii) and (iv), one of the three tetrahedra is the mirror image of the other two, and so his definition of space-filling is not satisfied. Although the tetrahedra of family (ii) are properly congruent, Sommerville argued that the prisms of this family cannot fill space without their mirror images.

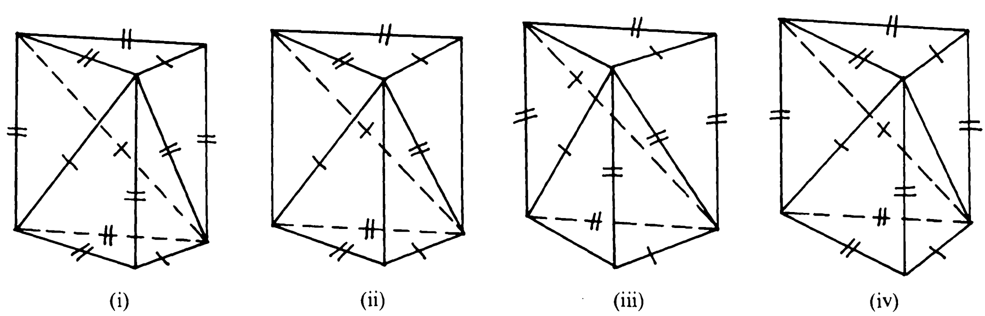
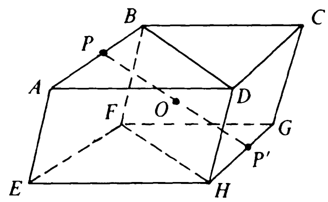


FIGURE 15. Schematic drawings of Sommerville's four families of triangular prisms which can be partitioned into congruent tetrahedra. There is only one member of the first family (up to similarity); the other families are infinite.

FIGURE 16.

A parallelopiped can be divided (in six different ways) into two mirror-image triangular prisms. The two prisms are related by inversion in the center *O*: if we join any point *P* of prism *ABDEFH* to *O* and extend this line segment by length $|OP|$, we find the corresponding point *P′* of the prism *BCDFGH*.



Every triangular prism fills space. The easiest way to see this is to note that any parallelopiped can be partitioned into two mirror-image triangular prisms (FIGURE 16). Conversely, any triangular prism can be joined to its mirror image upside-down along a parallelogram face to form a parallelopiped. (This may have been Haüy's reason for choosing parallelopipeds, tetrahedra, and triangular prisms for his basic polyhedral units.) If we partition a parallelopiped into prisms and then tetrahedra, and stack the parallelopipeds face-to-face, their constituent parts are juxtaposed with their mirror images. A triangular prism can be juxtaposed face-to-face with *properly* congruent copies only if each parallelogram face has mirror symmetry or if one does and the other two are mirror images. The prisms of Sommerville's second family do not satisfy either condition and this may be why he did not consider them to be true space-fillers. (M. Goldberg has pointed out that prisms of this family can fill space with directly congruent copies of the tetrahedra in a helix-like pattern, as shown on the cover. Stacking copies of such a prism end-to-end we obtain prisms of infinite length, the cross-sections of which are equilateral triangles. Then the infinite prisms can be packed together as in FIGURE 3. But this space-filling is not face-to-face.)

Even so, Sommerville's argument seems a little curious. In general, the tetrahedra derived from the prisms of family (ii) do not satisfy either of his necessary conditions (1) or (2) for face-to-face space filling. But in the special case when $a = d$, the tetrahedron is of the second kind. When copies of these tetrahedra are assembled into triangular prisms, they cannot fill space without their mirror images, but this fact does not prove that there is no way they can do so. By calculating dihedral angles one can show that in fact they do not fill space face-to-face, but this raises again the question Sommerville left unresolved, the status of the tetrahedra of this kind which cannot be obtained from a triangular prism.

On the other hand it is remarkable that, as far as we are aware, all the known space-filling tetrahedra, regardless of the technique used to find them, can be obtained from Sommerville's four prism families. H. S. M. Coxeter discussed three tetrahedra which generate space-filling copies by reflection in their faces [**15**, p. 84]; these turn out to be Sommerville's first and second, and the partition of the first shown in FIGURE 11(b). H. L. Davies rediscovered the tetrahedra of Sommerville's fourth prism family and obtained a second family by partition [**16**]. He also showed how Sommerville's first and fourth tetrahedra can be derived from these by specializing edge and angular relationships, and found another by partitioning the latter. L. Baumgartner discovered Sommerville's first, second, and fourth tetrahedra and an additional one obtained from the second by bisecting it with a plane containing a two-fold rotation axis [**13**], [**14**]. M. Goldberg, restricting himself to properly congruent tetrahedra, partitioned Sommerville's second family in the two possible ways to obtain three families [**18**]; as we have already noted, these space-fillings are not face-to-face. The five tetrahedra found by E. Koch in her computer study of a class of crystallographically important Dirichlet regions [**20**] are the four Sommerville tetrahedra plus the tetrahedron found by Baumgartner. Whether there are any tetrahedral space-fillers that cannot be obtained by partitioning a triangular prism remains an open question.

More generally, we can ask whether every tetrahedral space-filler can be obtained by partitioning a parallelohedron. We can also ask, as we suggested at the beginning of this section, whether there exist tetrahedra which fill space in an irregular way. (Indeed, it is possible that such tetrahedra might even satisfy Sommerville's conditions (a) and (b).) More than 2300 years after Aristotle, the question of which tetrahedra fill space and which do not is still unresolved!

The general space-filling problem is still wide open. Challenging open problems abound; the answers will be important not only for mathematics but also for crystallography and other fields concerned with the partition of space. We do not know the shapes of space-fillers, except for the parallelohedra and certain other special classes; we do not even know the maximum number of faces a space-filler can have, though the number has been proved to be finite for one important general class. The largest number of faces known to occur in a convex space-filler is thirty-eight, as was recently found by the crystallographer P. Engel [**17**]. Even more generally, there is the problem of filling space with copies of two or more kinds of polyhedra. In Plato's words, "their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey."

I would like to thank Branko Grünbaum, Susan Petrelli (Smith College, '82) and Lester Senechal for their helpful comments on the preliminary version of this paper, and Deedie Steele (Hampshire College, '81) for constructing excellent models of the Sommerville tetrahedra.

References

Suggestions for further reading are provided for each section, along with appropriate comments.

§1.

[1] Dirk J. Struik, De impletione loci, Nieuw Archief voor Wiskunde, 2nd series, 15(1925) 121–134. (Despite its Latin title, the article is written in Dutch; an English translation by the present author can be obtained from her or Prof. Struik, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02138. Struik's article contains many details and bibliographic references omitted in our exposition.)

[2] Dirk J. Struik, A Concise History of Mathematics, Dover, New York, 1948. (General background information about mathematics in this early period.)

[3] A. Beck, M. Bleicher, D. Crowe, Excursions into Mathematics, Worth, New York, 1969, pp. 21–30.

[4] B. Grünbaum and G. S. Shephard, Isohedral tilings of the plane by polygons, Comm. Math. Helv., 53(1978) 542–571. (A detailed discussion of many polygons which fill the plane in addition to the three regular ones cited here.)

[5] D. Schattschneider, Tiling the plane with congruent pentagons, this MAGAZINE, 51(1978) 29–44. (Discusses the special unsolved problem of tiling the plane with congruent irregular convex pentagons.)

§2.

[6] John C. Burke, Origins of the Science of Crystals, Univ. of California Press, Berkeley and Los Angeles, 1966. (Describes the history of crystallography up to about 1850.)

[7] L. Fejes Tóth, Regular Figures, Pergamon Press, New York, 1965, pp. 114–119, 121–123.

[8] Alan Holden and Phyllis Singer, Crystals and Crystal Growing, Science Study Series, Anchor Books, Garden City, NY, 1960. (A very readable account of crystals and their properties.)

[9] Johannes Kepler, The Six-Cornered Snowflake, Latin text and English translation, Oxford Univ. Press, 1966.

[10] Marjorie Senechal, A brief history of geometric crystallography, to appear in A Historical Atlas of Crystallography, J. Lima de Faria (Editor), International Union of Crystallography. (Describes the history of geometric crystallography up to the present.)

[11] Peter Pearce, Structure in Nature Is a Strategy for Design, MIT Press, 1978.

[12] Robert Williams, The Geometric Foundations of Natural Structure: A Source Book of Design, 2nd ed., Dover, New York, 1979. (The most detailed of several books in which space-filling polyhedra are discussed; another is [**11**].)

§3.

[13] L. Baumgartner, Zerlegung des vierdimensionalen Raumes in kongruente Fünfzelle, Mathematische-Physikal Semesterberichte, 15(1968) 76–86.

[14] ______, Zerlegung des n-dimensionalen Raumes in kongruente Simplexe, Mathematische Nachrichten, 48(1971) 213–224.

[15] H. S. M. Coxeter, Regular Polytopes, 3rd ed., Dover, New York, 1973.

[16] H. L. Davies, Packings of spherical triangles and tetrahedra, Proceedings of the Colloquium on Convexity, Copenhagen, 1965, pp. 42–51.

[17] P. Engel, Über Wirkungsbereichsteilungen von kubischer Symmetrie, Z. Krist., 154(1981) 199–215.

[18] M. Goldberg, Three infinite families of tetrahedral space-fillers, J. Combin. Theory, 16(1974) 348–354.

[19] B. Grünbaum and G. S. Shephard, Tilings with congruent tiles, Bull. Amer. Math. Soc., New Series, 3(1980) 951–973. (Gives a survey of the general problem of filling space with congruent shapes.)

[20] E. Koch, Wirkungsbereichspolyeder und Wirkungsbereichsteilungen zu kubischen Gitterkomplexen mit weniger als drei Freiheitsgraden, Dissertation, Philipps-Universität, Marburg/Lahn, 1972.

[21] D. M. Y. Sommerville, Division of space by congruent triangles and tetrahedra, Proc. Royal Society of Edinburgh, 43(1923) 85–116.

[22] ______, Space-filling tetrahedra in Euclidean space, Proc. Edinburgh Math. Soc., 41(1923) 49–57.

NOTES

Rearranging Terms in Alternating Series

RICHARD BEIGEL
Stanford University
Stanford, CA 94305

It is well known that if you change the order of the terms in a conditionally convergent series, the result may have a different sum. We will demonstrate a method for calculating the sum of a broad class of such reordered series. The result is not new; it originally appeared in a 19th century German paper [**8**], and similar work appeared later in [**2**], [**4**], and [**6**]. The purpose of this note is to bring attention to the result, which could appropriately be mentioned in introductory calculus classes right after Riemann's theorem [**1**]; the main result uses nothing more complicated than the integral test, and the discussion uses nothing more complicated than the concept of $o(\frac{1}{x})$, which could be explained (see [**5**]) or circumnavigated in lecture. The results from [**3**] on the alternating harmonic series could well be included in the lecture.

Let $S=\sum_{j=1}^{\infty}(-1)^{j+1}a_j$, with $a_j>0$ and $\{a_j\}$ eventually monotonically decreasing to zero. Define S^{mn} as the rearrangement (no signs changed) obtained from S by taking groups of m positive terms followed by groups of n negative terms (thus $S^{11}=S$ and $S^{21}=a_1+a_3-a_2+a_5+a_7-a_4+\cdots$).

THEOREM. *Let f be a continuous and eventually monotone function such that $f(2k-1)=a_{2k-1}$ for positive integral k, and let $m\geqslant n>0$. Then*

$$S^{mn}=S+\frac{1}{2}\lim_{k\to\infty}\int_{nk}^{mk}f(x)\,dx,$$

with the understanding that both sides may be infinite or may diverge by oscillation.

Proof. Since $a_k\to 0$, S^{mn} is the limit of its $(m+n)k$th partial sum. Identically,

$$S_{(m+n)k}^{mn}=S_{2nk}+\sum_{j=nk+1}^{mk}a_{2j-1}. \tag{1}$$

Since f is eventually monotone decreasing to zero, the limit

$$\lim_{t\to\infty}\left(\sum_{j=1}^{t}f(2j-1)-\int_1^t f(2x-1)\,dx\right)$$

exists; this follows from the standard proof of the integral test for convergence. It follows that the tail expression goes to 0; therefore,

$$\lim_{k\to\infty}\left(\sum_{j=nk+1}^{mk}a_{2j-1}-\int_{nk}^{mk}f(2x-1)\,dx\right)=0.$$

Therefore,

$$\lim_{k\to\infty}\sum_{j=nk+1}^{mk}a_{2j-1}=\lim_{k\to\infty}\int_{nk}^{mk}f(2x-1)\,dx$$

$$= \lim_{k\to\infty} \frac{1}{2}\int_{2nk-1}^{2mk-1} f(x)\,dx$$

$$= \frac{1}{2} \lim_{k\to\infty} \int_{nk}^{mk} f(x)\,dx, \qquad (2)$$

again with the understanding that both limits may be infinite or may diverge by oscillation. Let $k \to \infty$ in (1) and use (2) to finish the proof.

REMARK. If $\{a_j\}$ is absolutely convergent, then we get the expected result that $S^{mn} = S$. If $m = n$ we also find, as expected, that $S^{mn} = S$. (Several less obvious sum-preserving rearrangements of infinite series are discussed in [7].)

Several examples will illustrate the diverse results that can be obtained by applying our Theorem to rearrangements S^{mn} of particular alternating series.

EXAMPLE 1. If H^{mn} is obtained from the alternating harmonic series $(1 - 1/2 + 1/3 - 1/4 + \cdots)$ by taking groups of m positive terms and n negative terms, then

$$H^{mn} = \log 2 + \frac{1}{2} \lim_{k\to\infty} \int_{nk}^{mk} \frac{1}{x}\,dx = \log 2 + \frac{1}{2}\log\frac{m}{n} = \frac{1}{2}\log\frac{4m}{n}.$$

EXAMPLE 2. If P^{mn} is obtained similarly from the alternating series $1 - 1/2^p + 1/3^p - 1/4^p + \cdots$, with $0 < p < 1$ and $m > n$, then P^{mn} diverges to positive infinity.

EXAMPLE 3. Let L be Leibniz's sum for $\pi/4$: $L = 1 - 1/3 + 1/5 - 1/7 + \cdots$. Then

$$L^{mn} = \frac{\pi}{4} + \frac{1}{2} \lim_{k\to\infty} \int_{nk}^{mk} (1/(2x-1))\,dx = \frac{\pi}{4} + \frac{1}{4}\log\frac{m}{n}.$$

EXAMPLE 4. Let $S = \sum_{j=1}^{\infty}(-1)^{j+1}/j\log j$. Then

$$S^{mn} = S + \frac{1}{2} \lim_{k\to\infty} (\log\log mk - \log\log nk) = S.$$

EXAMPLE 5. Let $S = \sum_{j=1}^{\infty}(-1)^{j+1} f(j)$, where $f(x) = (\cos(\log x) + 2)/x$. It is easy to check that f is monotone.

$$S^{mn} = S + \tfrac{1}{2} \lim_{k\to\infty} (\sin(\log mk) + 2\log mk - \sin(\log nk) - 2\log nk),$$

which diverges by oscillation unless $m = n$.

The curious reader may be wondering if we can find an example for which S^{mn} converges to a value differing from S by something other than a logarithmic term. Barring "artificial" functions for which $\lim_{x\to\infty} x f(x)$ diverges by oscillation, there is none.

Proof. If $f(x) = o(1/x)$, then

$$\lim_{k\to\infty} \int_{nk}^{mk} f(x)\,dx \leq \lim_{k\to\infty} (mk - nk) f(nk), \text{ since } f \text{ is monotone decreasing,}$$

$$= \lim_{k\to\infty} \frac{m-n}{n} nk f(nk),$$

$$= 0.$$

Therefore $S^{mn} = S$. If $f(x) = c/x + o(1/x)$, then

$$\lim_{k\to\infty} \int_{nk}^{mk} f(x)\,dx = \lim_{k\to\infty} \int_{nk}^{mk} \frac{c}{x} + o\left(\frac{1}{x}\right) dx$$

$$= \lim_{k\to\infty} \int_{nk}^{mk} \frac{c}{x}\,dx, \text{ by the preceding result,}$$

$$= c\log\frac{m}{n},$$

so $S^{mn} = S + \frac{1}{2}c\log(m/n)$. The reader may check that S^{mn} diverges to positive infinity if $f(x) > O(1/x)$.

The remaining possibility, that $xf(x)$ diverges by oscillation, seems to merit attention. Suppose we modify Example 5 slightly by letting $f(x) = (\cos(\alpha\log x) + 2)/x$, where $\alpha = 2\pi/\log(m/n)$. Then

$$\alpha\log mk - \alpha\log nk = \alpha\log\frac{m}{n} = 2\pi, \text{ so } \sin(\alpha\log mk) - \sin(\alpha\log nk) = 0.$$

Hence

$$S^{mn} = S + \frac{1}{2}\lim_{k\to\infty}\left(\sin(\alpha\log mk) + 2\log mk - \sin(\alpha\log nk) - 2\log nk\right) = S + \log\frac{m}{n}.$$

Once again the difference is proportional to $\log(m/n)$. Also, notice that S^{pq} converges if and only if $\alpha\log(p/q)$ is an integral multiple of 2π, or equivalently if and only if p/q is an integral power of m/n. It would be interesting to find answers to the following two questions:

Is there a series S for which $S^{mn} - S$ converges for some values of m and n, but is not proportional to $\log(m/n)$?

Is there a series $S = \sum_{j=1}^{\infty}(-1)^{j+1}f(j)$ such that $xf(x)$ diverges by oscillation but $S^{mn} - S$ converges for all values of m and n?

EXAMPLE 6. Suppose $S = \sum_{j=1}^{\infty}(-1)^{j+1}\sin(1/j)$. Naturally we take $f(x)$ to be $\sin(1/x)$, which is $1/x + o(1/x)$. Thus we find that

$$S^{mn} = S + \frac{1}{2}\log\frac{m}{n}\lim_{x\to\infty} x\sin\frac{1}{x} = S + \frac{1}{2}\log\frac{m}{n}.$$

Getting back to our theorem, we should note that the assumption that $m \geqslant n$ is unnecessary—to see this, just identify $\sum_{j=nk+1}^{mk} a_{2j-1}$ with $-\sum_{j=mk+1}^{nk} a_{2j-1}$. In fact, a slight modification (left to the reader) of our proof yields a similar result for any rearrangement of S that leaves the order of its positive and negative subsequences intact; if S' is such a rearrangement of S with $\phi(k)$ negative terms in S'_k, and f satisfies our old hypothesis, then

$$S' = S + \frac{1}{2}\lim_{k\to\infty}\int_{\phi(k)}^{k-\phi(k)} f(x)\,dx.$$

Applying this result to the series $S = 1 - 2^{-1/2} + 3^{-1/2} - 4^{-1/2} + \cdots$, and taking $\phi(k)$ to be the integral part of $((l - \sqrt{2k - l^2})/2)^2$ for $k > \frac{1}{2}l^2$ (ϕ can be arbitrary elsewhere), a simple calculation shows that the rearranged series sums to $S + l$.

For another application of this result, choose the reordering, H', of the alternating harmonic series for which $\phi(k)$ equals the integral part of $k/(\frac{1}{4}e^{2\pi} + 1)$. Then $H' = \pi$.

I wish to express my thanks to all those who read over and commented on this manuscript, especially Alan Siegel.

References

[1] T. Apostol, Calculus, vol. I, 2nd ed., Blaisdell, MA, 1967, p. 413.

[2] T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., Macmillan, London, 1947, pp. 74-76.

[3] C. C. Cowen, K. R. Davidson, and R. P. Kaufman, Rearranging the alternating harmonic series, Amer. Math. Monthly, 87 (1980) 817–819.

[4] K. Knopp, Theorie und Anwendung der unendlichen Reihen, 5th ed., Springer-Verlag, Berlin, 1964, p. 335 (problem 148).

[5] J. Olmsted, Advanced Calculus, Appleton-Century-Crofts, NY, 1961, p. 141.

[6] A. Pringsheim, Ueber die Werthveränderungen bedingt convergenter Reihen und Producte, Math. Ann., 22(1883) 455–503.

[7] Paul Schaefer, Sum-preserving rearrangements of infinite series, Amer. Math. Monthly, 88(1981) 33–40.

[8] O. Schlömilch, Ueber bedingt-convergirende Reihen, Z. Math. Phys., 18(1873) 520–522.

A Model for Playing Time

DAN KALMAN
University of Wisconsin–Green Bay
Green Bay, WI 54302

Most audio recording devices (reel-to-reel, cassette, and cartridge tape recorders) are equipped with a numerical counter which is incremented by the advance of the tape, and which enables the user to compile a table of contents for the tape. In many instances it is desired to relate the number displayed by the counter (hereafter **counter number**) to time; for example, to determine the **playing time** of a selection. In this note I will describe the development of a model relating counter number and time for the case in which the counter number, c, varies directly with the angular displacement, θ, of the supply reel. (The hypothesis that c and θ are proportional was formed after observing the operation of a cassette recorder in forward and reverse with no tape in place. This hypothesis was confirmed when, for 11 data points, the ratio of counter number to number of supply reel revolutions had an average deviation of .006 from the mean ratio of .527. For some tape recorders the hypothesis might fail.)

The equation which will be derived is

$$t=\frac{\pi}{ms}\left[2r_0c-\frac{(r_0-r_1)}{c_1}c^2\right],$$

which relates time, t, and counter number, c. The parameters in this equation are r_0, the initial radius of the supply reel; r_1 and c_1, simultaneously observed values of radius and counter number at some time after beginning the tape; s, the speed of the tape; and m, the proportionality factor relating c and the number of revolutions of the supply reel.

The primary fact used in modeling the tape recorder system is that the tape is drawn past the recording heads at a constant linear speed, s, causing, in turn, the motion of the supply reel. This motion can be described in terms of l, the length of tape drawn past the head; r, the radius of the spool of tape; and θ, all of which vary with t. FIGURE 1 is a schematic representation of the tape recorder system depicting the variables $r(t)$, $l(t)$, and $\theta(t)$, as well as the counter number $c(t)$. It

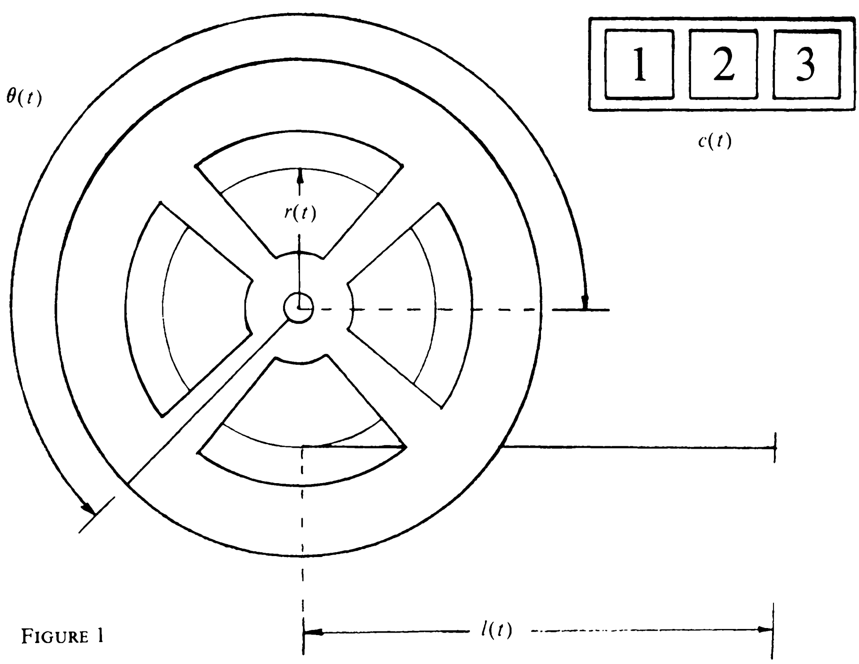


FIGURE 1

is assumed that initially $r = r_0$ and all other variables equal zero.

Our model assumes $dl/dt = s$ is a constant. The rate of change of $l(t)$ can be related to θ and r as follows. A change $\Delta\theta$ in the angle increases l by the arc length of the outermost wind of tape between θ and $\theta + \Delta\theta$. For $\Delta\theta$ small, we see from FIGURE 2 that if the variation of r with t is neglected, this arc length is

$$\Delta l = r\Delta\theta$$

from which it follows that

$$\frac{\Delta l}{\Delta t} = r\frac{\Delta\theta}{\Delta t}.$$

This suggests that, in the limit

$$\frac{dl}{dt} = r\frac{d\theta}{dt}$$

and so

$$r\frac{d\theta}{dt} = s. \tag{1}$$

Turning to r, note that an increase of 2π in θ removes one layer of tape from the reel and decreases r by τ, the thickness of the tape. Assuming this decrease is distributed evenly throughout the wind gives

$$\Delta r = -\tau\frac{\Delta\theta}{2\pi}$$

for $\Delta\theta$ less than 2π, leading to

$$\frac{dr}{dt} = \frac{-\tau}{2\pi}\frac{d\theta}{dt}. \tag{2}$$

Together, (1) and (2) form a system of differential equations that can be solved by elementary means. In particular, knowledge of the chain rule and the general solution of $f'(t) = 0$ provide a sufficient background for understanding a solution of this system. With the initial condition, θ may be explicitly formulated as a function of t, and this relation inverted to give t in terms of θ, thus:

$$t = \frac{r_0}{s}\theta - \frac{\tau}{4\pi s}\theta^2. \tag{3}$$

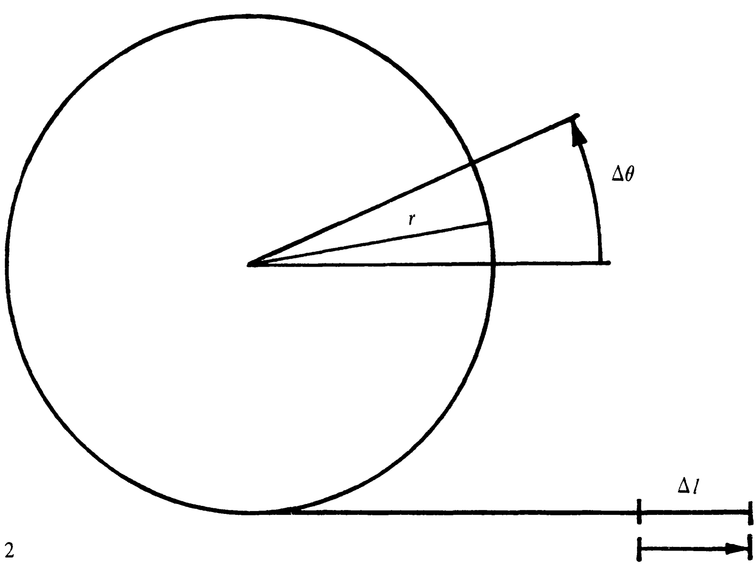


FIGURE 2

Notice that (3) imposes a maximum value on t when $\theta = 2\pi r_0/\tau$. This is consistent with what we would expect since $\theta/2\pi$, the number of revolutions of the reel, can't exceed r_0/τ, the number of winds of tape. Actually, there is a central hub of the reel of radius r_h and only $(r_0 - r_h)/\tau$ winds of tape. Thus Equation (3) is only valid for

$$0 \leq \theta \leq \frac{2\pi(r_0 - r_h)}{\tau} < \frac{2\pi r_0}{\tau}.$$

Interpreting $\theta/2\pi$ as the number of revolutions also allows the introduction of c in (3). If each revolution increments the counter by m, then $c = m\cdot\theta/2\pi$. Now θ is eliminated from (3) with the result

$$t = \frac{\pi}{ms}\left(2r_0c - \frac{\tau}{m}c^2\right). \tag{4}$$

One of the virtues of this problem is the ease with which the model may be empirically tested. This is, of course, a vital part of modeling, but one which must often be omitted in mathematics courses. As already mentioned, m was determined to be .527 for my recorder. The electronics industry was consulted for values of s and τ; they were 1-7/8 inches per second (112.5 inches per minute) and one-half mill (5×10^{-4} inches), respectively, for my trial. For r_0, I resorted to a crude measurement by eye and ruler, resulting in a value of 1 inch. This measurement might easily have involved an error of 1/8 inch, or so. Substitution of these values in (4) and rounding coefficients to two significant figures produced the equation $t = .11c - 5 \times 10^{-5}c^2$ (minutes). Finally, data were acquired to test this last equation for accuracy by allowing the tape recorder to play while watching the sweep hand of a clock, and recording the counter values at 30 or 60 second intervals. For the first tape I tried, I recorded 88 data points spanning the entire length of the tape. On the average, the absolute deviation between the predicted and observed values of t was .02 minutes; very good agreement, indeed! (In fact, a least squares fit of an equation of type (4) to the data reproduced the theoretical coefficients to two significant figures.) However, successive trials with other tapes fared much worse, producing large discrepancies between predicted and observed values of t. Evidently, the success of the first trial was not indicative of the accuracy of the model in general, and some revision was needed.

The rationale for revising the model involves an additional detail about the operation of the counter. When recording or playing tapes, I usually zero the counter with the tape completely rewound. Yet later, when the tape is again rewound, the counter rarely returns to zero. This suggests that the tape is wrapped tighter under some conditions than others. Accordingly, the thickness of each layer of tape on the spool is not just the thickness of the tape, but includes some air space as well. This observation leads to a new interpretation of the parameter τ.

After some experimentation, it was determined that completely advancing and rewinding the tape at high speed three or four times achieves a stabilization in the tightness of the wrap. Assuming now that the tightness is uniform within the spool, the combined thickness of air space and tape can be determined for each tape and used in place of the constant τ. To determine this combined thickness, a change in the radius of the spool is divided by the number of winds removed from the spool. Since each wind is an increment of m in c, the number of winds is $\Delta c/m$, and the combined thickness is

$$\tau_c = \frac{m\Delta r}{\Delta c}.$$

Now, from the initial conditions when $c = 0$, advance the tape to a new point and obtain r_1 and c_1 there. This gives $\tau_c = m(r_0 - r_1)/c_1$ and substitution in (4) simplifies to

$$t = \frac{\pi}{ms}\left[2r_0c - \frac{(r_0 - r_1)}{c_1}c^2\right]. \tag{5}$$

I tried out (5) on three tapes. For each one I advanced and rewound the tape several times before measuring r_0 and zeroing the counter. The tape was allowed to play while a stopwatch displayed elapsed time, and several data points were gathered. When the last point was observed, the tape was stopped and r_1 was also determined. The results of these tests are summarized in TABLE 1.

Tape Number	Number of Observations	Greatest Deviation (minutes)	Mean Deviation (minutes)
1	11	.15	.072
2	8	1.00	.37
3	9	.35	.17

TABLE 1. **Summary of absolute deviations between observed times and predicted times.**

Because I was required to advance and rewind each tape several times and gather an additional reading of c and r, it took between five and ten minutes to determine the lengths of the separate selections on each tape using the model. In itself, this is a dramatic savings over the time it would have required without a model. However, considering the time spent in developing the model, I'm not sure there was an overall savings of time. On the other hand, I am sure that developing the model was a more productive use of time, and it was much more fun.

Making Connections: A Graphical Construction

DOROTHY WOLFE
Widener College
Chester, PA 19013

Suppose you have a list of club members who would like to exchange letters with one or more other members. The most active member tells you he wants 12 pen pals, the next two members want 11 pen pals each, and so on. Each member requests a specified number of pen pals. When you check up, you find that you have 74 letters to send out, and they've come to you from the 16 people on your list in the following batches: 12, 11, 11, 5, 5, 5, 5, 5, 4, 4, 2, 1, 1, 1, 1, 1.

In terms of number theory this sequence represents a **partition** of the integer 74, a collection of positive integers adding up to 74. (See [**2**] or [**3**].) Your problem is to make sure that each person receives as many letters as he's sent out, that is, to find an appropriate **graph**, a set of points called **vertices** (representing the members) with connecting lines called **edges** between some of them (representing a unique exchange of letters between a pair).

The **degree** of a vertex of a graph is the number of edges extending from it. A list consisting of the degrees of each vertex of a graph is called a **degree sequence**. Not all sequences of integers can be degree sequences of graphs. For example, there is no graph whose vertices have degrees 4, 2, 2, 2. (You might try to draw such a graph; you will find that there will be only three vertices for the four edges from the first vertex to meet.) Nor will 4, 2, 2, 2, 1 do (since each edge is counted twice, the sum of the numbers in a degree sequence must be even). Thus, given the original problem, you might not be able to find a graph with a degree sequence that solves it.

The question of whether a given partition is the degree sequence of a graph is answered by a well-known theorem of Erdős and Gallai [**1**]. The Erdős-Gallai condition gives the criterion in combinatorial terms that (roughly) ensure that early numbers in the partition are not too large in displayed elapsed time, and several data points were gathered. When the last point was observed, the tape was stopped and r_1 was also determined. The results of these tests are summarized in TABLE 1.

Tape Number	Number of Observations	Greatest Deviation (minutes)	Mean Deviation (minutes)
1	11	.15	.072
2	8	1.00	.37
3	9	.35	.17

TABLE 1. **Summary of absolute deviations between observed times and predicted times.**

Because I was required to advance and rewind each tape several times and gather an additional reading of c and r, it took between five and ten minutes to determine the lengths of the separate selections on each tape using the model. In itself, this is a dramatic savings over the time it would have required without a model. However, considering the time spent in developing the model, I'm not sure there was an overall savings of time. On the other hand, I am sure that developing the model was a more productive use of time, and it was much more fun.

Making Connections: A Graphical Construction

DOROTHY WOLFE
Widener College
Chester, PA 19013

Suppose you have a list of club members who would like to exchange letters with one or more other members. The most active member tells you he wants 12 pen pals, the next two members want 11 pen pals each, and so on. Each member requests a specified number of pen pals. When you check up, you find that you have 74 letters to send out, and they've come to you from the 16 people on your list in the following batches: 12, 11, 11, 5, 5, 5, 5, 5, 4, 4, 2, 1, 1, 1, 1, 1.

In terms of number theory this sequence represents a **partition** of the integer 74, a collection of positive integers adding up to 74. (See [2] or [3].) Your problem is to make sure that each person receives as many letters as he's sent out, that is, to find an appropriate **graph**, a set of points called **vertices** (representing the members) with connecting lines called **edges** between some of them (representing a unique exchange of letters between a pair).

The **degree** of a vertex of a graph is the number of edges extending from it. A list consisting of the degrees of each vertex of a graph is called a **degree sequence**. Not all sequences of integers can be degree sequences of graphs. For example, there is no graph whose vertices have degrees 4, 2, 2, 2. (You might try to draw such a graph; you will find that there will be only three vertices for the four edges from the first vertex to meet.) Nor will 4, 2, 2, 2, 1 do (since each edge is counted twice, the sum of the numbers in a degree sequence must be even). Thus, given the original problem, you might not be able to find a graph with a degree sequence that solves it.

The question of whether a given partition is the degree sequence of a graph is answered by a well-known theorem of Erdős and Gallai [1]. The Erdős-Gallai condition gives the criterion in combinatorial terms that (roughly) ensure that early numbers in the partition are not too large in relation to later numbers. In this note we offer another criterion; it accomplishes the same thing in number-theoretic terms. We shall illustrate how to use our criterion by constructing a graph with the partition of 74 in our problem as its degree sequence.

With each partition $(d_1, d_2, \ldots, d_q)$ with $d_i \geqslant d_j > 0$ for $i < j$, we can associate an array of dots called its **Ferrers diagram** [**3**]. This is a set of dots aligned in rows and columns, both in nonincreasing order, so that the ith row has d_i dots and all rows begin with a dot in the first column. If we let a dot represent a letter sent out, we'll get the Ferrers diagram F for the original problem, shown in FIGURE 1.

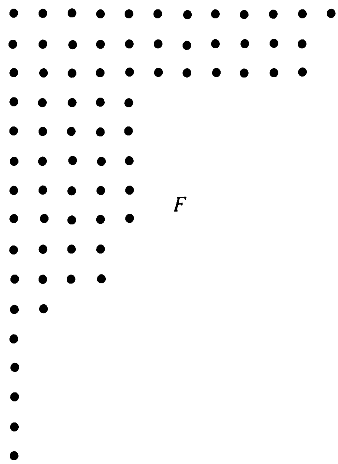


FIGURE 1

Counting the number of dots in each column produces another partition of 74, called the **conjugate partition**: 16, 11, 10, 10, 8, 3, 3, 3, 3, 3, 3, 1. As a first attempt to assign pen pals, you might label the columns with names of members in the same order as the rows, then send each letter out to the person whose name heads the column in which it is represented. The column totals will tell you how many letters each member receives. But that won't do. No member wants to receive a letter from himself, so you'll certainly have to remove dots from the diagonal that starts at the top left. You'll have to change the column totals—but not those of the rows—to make sure that each person receives as many letters as he's sent. The result, with a minor change of notation, will be the **adjacency matrix** of a graph, a symmetric matrix of zeros and ones, with $a_{i,i} = 0$, $a_{i,j} = 0$ if vertices p_i and p_j are not connected by edges, and $a_{i,j} = 1$ if p_i and p_j are connected by edges.

We shall construct a transformation from Ferrers diagram to adjacency matrix. In doing this we'll prove the following theorem, which contains our criterion for a partition to be the degree sequence of a graph.

THEOREM. *Let D be a partition of an even integer $2n$, $D = (d_1, \ldots, d_q)$ with $d_1 \geqslant \cdots \geqslant d_q > 0$ having conjugate partition $K = (k_1, \ldots, k_r)$, $k_1 \geqslant \cdots \geqslant k_r > 0$. Let h be the largest integer such that $d_h \geqslant h$. D is the degree sequence of a graph if and only if for each r such that $1 \leqslant r \leqslant h$ the following inequality holds*:

$$\sum_1^r d_i \leqslant \sum_1^r (k_i - 1). \tag{1}$$

The corner square of h^2 dots in the Ferrers diagram is called the **Durfee square** [**2**]; it is the largest subsquare of dots in the Ferrers diagram. Note that $d_{h+1} < h+1$, $k_h \geqslant h$, and $k_{h+1} < h+1$. In FIGURE 1, $h = 5$.

The above theorem can be proved using the Erdős-Gallai theorem [**4**]; in fact the two theorems are equivalent. The Erdős-Gallai condition is that for all r, the following inequality holds:

$$\sum_{1}^{r} d_i \leqslant r(r-1) + \sum_{r+1}^{q} \min\{r, d_i\}.$$

Our proof is a constructive one; our theorem could provide an alternative proof of the Erdős-Gallai criterion.

Proof. It is easy to prove that condition (1) is necessary. For suppose there exists a graph having D as the degree sequence. The vertices with greatest degree will have as labels the smallest numbers. The adjacency matrix A of the graph has d_i as the sum both of its ith row and of its ith column. (In either case, this is the number of edges incident with the vertex p_i.) We can construct a new matrix B from A (B will look like a Ferrers diagram if we identify the 1's with dots and 0's with blanks) as follows. To form the ith row of B, shift the positions of the 1's and 0's in the ith row of A so that all of the 1's are now placed in the first d_i columns and the 0's fill out the rest of the row. Clearly B has the same row sums as A, but the column sums are almost surely different. To examine the column sums, we suppose that the transformation from A to B is carried out in two stages. First, note that for $i \leqslant h$ it is true that $d_i \geqslant d_h \geqslant h$, so there must be a 1 at $a_{i,h}$ or to its right, and this 1 can be moved left to $a_{i,i}$. If we let c_i be the ith column sum of the matrix after this shift for all $i \leqslant h$, then for such i, we have $c_i \geqslant d_i + 1$ and so for $r \leqslant h$,

$$\sum_{1}^{r} (d_i + 1) \leqslant \sum_{1}^{r} c_i. \tag{2}$$

After this first shift, we move all 1's in each row as far as possible to the left to obtain the matrix B. If we let k_i be the ith column sum of B, then for all r

$$\sum_{1}^{r} c_i \leqslant \sum_{1}^{r} k_i. \tag{3}$$

Using (2) and (3), we see that for $r \leqslant h$

$$\sum_{1}^{r} d_i \leqslant \sum_{1}^{r} (k_i - 1),$$

so that (changing 1's to dots) B provides a Ferrers diagram with the required property.

To prove the sufficiency of (1) we will transform a Ferrers diagram F to an adjacency matrix, and this requires a bit more work. Our problem is to move dots to the right so that the number of dots in each row remains the same and columns become symmetric with rows. One way to do this would be to use the Havel-Hakimi process ([**1**], Chap. 6). The construction below is better adapted to make use of our condition (1). It is a process by which we construct one row and column of an adjacency matrix at a time, with the resulting adjacency matrix corresponding to a graph which has D as its degree sequence. There are two cases, requiring two different strategies. First, if inequality (1) becomes an equality for any $r \leqslant h$, we work with the diagram F to eliminate row r. (If there is more than one such r, choose the largest.) Second, if inequalities (1) are all strict, we begin with row h, the last row such that $d_h \geqslant h$.

The pen pal problem and its Ferrers diagram in FIGURE 1 were designed to satisfy condition (1) in order to illustrate the various situations that might arise in each case. In this example $h = 5$ and $q = 16$. For convenience we shall refer to the intersection of the ith row and jth column of the Ferrers diagram F as position $a_{i,j}$. In altering F, we shall always *move dots to the right without changing the number of dots in any row*. After altering the placement of dots in a given row of F and if necessary, increasing the number of dots in the corresponding column, we then substitute 1's for dots and 0's for blanks to fill in a 16×16 adjacency matrix.

In FIGURE 1, the only $r \leqslant h$ for which the two sides of (1) are equal is $r = 3$. The left side becomes $\Sigma_1^3 d_i = 34$ and the right side is $\Sigma_1^3 (k_i - 1) = 37 - 3 = 34$. To fill in row 3 and column 3 of the adjacency matrix, we first empty the diagonal position of F: the dot at $a_{3,3}$ is moved to the first empty position on the right end of the row, which is $a_{3,12}$. Row 3 has 11 dots and column 3 now has 9 dots, so we must increase the number of dots in column 3 by 2. To do this, we will

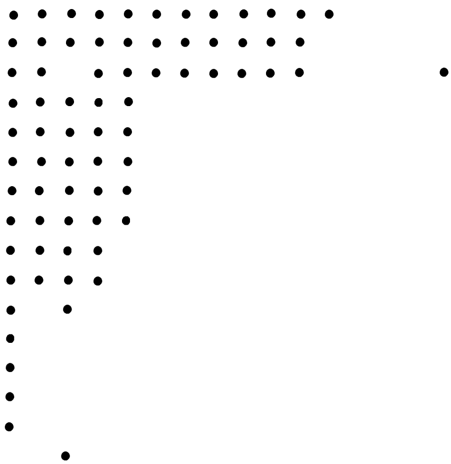


FIGURE 2

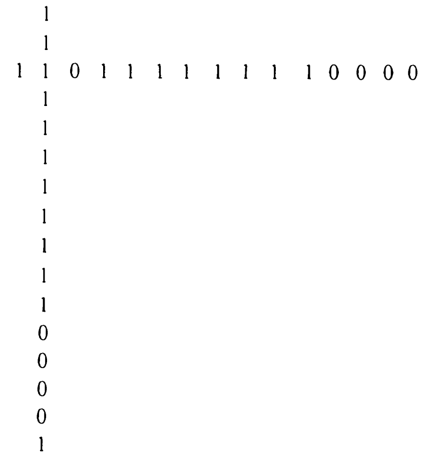


FIGURE 3

move rightward to column 3 two dots chosen from below the tenth row in columns to the left of column 3. Although inspection of the diagram in FIGURE 1 shows that dots in column 1 and 2 below row 10 are available to be moved (right) to column 3, in the general case, it is condition (1) that guarantees the existence of these dots, available to be moved in a similar manner, to create a symmetric row and column of dots. In our case, we first move one dot from $a_{11,2}$ to $a_{11,3}$. Since no other dots occur in column 2 below row 11, we look to the next column to the left (column 1) and move a dot in any row below row 11. We arbitrarily choose the lowest and shift a dot from $a_{16,1}$ to $a_{16,3}$. Finally we make row 3 symmetric with column 3; we move dot $a_{3,12}$ to $a_{3,16}$. The new configuration of dots we have obtained from F is shown in FIGURE 2. We use this configuration both to obtain row 3 and column 3 of the adjacency matrix we seek to construct and to produce a new Ferrers diagram to continue the construction. Replacing dots by 1's and blanks by 0's in row 3 and column 3 of FIGURE 2 results in the corresponding row and column of the adjacency matrix under construction (see FIGURE 3). Striking out row 3 and column 3 in the configuration of FIGURE 2 produces the configuration in FIGURE 4(a); from this we can obtain a new Ferrers diagram F^* shown in FIGURE 4(b) (just eliminate spaces so as to abide by Ferrers diagram rules). F^* is the Ferrers diagram of a degree sequence D^* of length $q^* < q$ (in our example, $q^* = 14 = q - 2$), and the size h^* of the Durfee square in F^*, is $h - 1$. (This is because the dots eliminated from F were from row 3 and column 3.) If we let K^* be the conjugate partition of D^*, then it is an easy exercise to show that F^* satisfies the conditions of the theorem for $1 \leqslant r \leqslant h^*$.

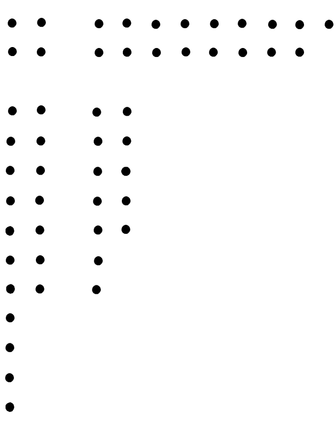


FIGURE 4 (a)

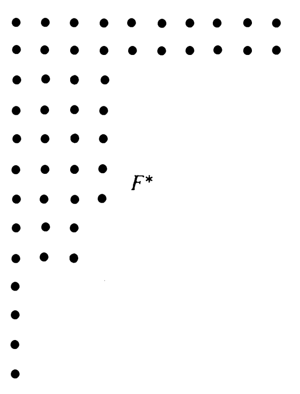


FIGURE 4 (b)

0	1	1	1	1	1	1	1	1	1	0	1	1	1	0	0
1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	0
1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	1
1	1	1													
1	1	1													
1	1	1													
1	1	1													
1	1	1													
1	1	1													
1	1	1													
1	1	1													
0	1	1													
1	0	0													
1	0	0													
1	0	0													
0	1	0													
0	0	1													

FIGURE 5

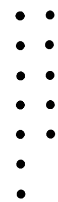


FIGURE 6

The process is repeated with F^*. In every case, if we start with row r in F for which the sums in (1) are equal, equality will hold for sums of $r-1$ terms in F^*, and so it is in our example. For $r=2$ in our new diagram, $d_1^* + d_2^* = 21$ and $k_1^* + k_2^* = 23$. We work on row 2 and column 2 as before, then repeat the process with row 1 and column 1. We have progressed in matrix building to FIGURE 5.

At this stage we have struck out all dots in the first three rows and columns of our original Ferrers diagram F, and all that remains is shown in FIGURE 6. This diagram is to be transformed into a 13×13 adjacency matrix, filling in the southeast corner of FIGURE 5.

The Ferrers diagram in FIGURE 6 gives us a chance to show the strategy of our construction when all inequalities (1) of the theorem are strict. In this case we must work on row h, the row with the largest subscript such that $d_h \geq h$. If we look at it as a fresh problem, our diagram in FIGURE 6 has $h=2$, $d_1 = d_2 = 2$, $k_1 = 7$, and $k_2 = 5$. So (1) is a strict inequality for $r=1$ and $r=2$. Again we need $a_{i,i} = 0$, and so dot $a_{2,2}$ of our example is moved to position $a_{2,3}$. If $k_2 < d_2$ we proceed as in the first case and move dots rightward to column 2, then adjust the position of dots in row 2 to make row 2 and column 2 symmetric. Since $k_2 > d_2$, we instead retain $k_2 = 2$ dots at the bottom of column 2 (i.e., dots $a_{5,2}$ and $a_{4,2}$ retain their positions) and move all dots in column 2 above these one position to the right. Now to make row 2 symmetric with column 2, we move dot $a_{2,3}$ to position $a_{2,5}$ and dot $a_{2,1}$ to position $a_{2,4}$. The adjusted diagram is shown in FIGURE 7; we use this in the same way that the configuration in FIGURE 2 was used earlier.

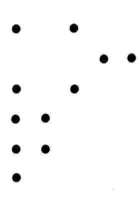


FIGURE 7

0	1	1	1	1	1	1	1	1	1	0	1	1	1	0	0
1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	0
1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	1
1	1	1		0											
1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0
1	1	1		0											
1	1	1		1											
1	1	1		1											
1	1	1		0											
1	1	1		0											
0	1	1		0											
1	0	0		0											
1	0	0		0											
1	0	0		0											
0	1	0		0											
0	0	1		0											

FIGURE 8

0	1	1	1	1	1	1	1	1	1	0	1	1	1	0	0
1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	0
1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	1
1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

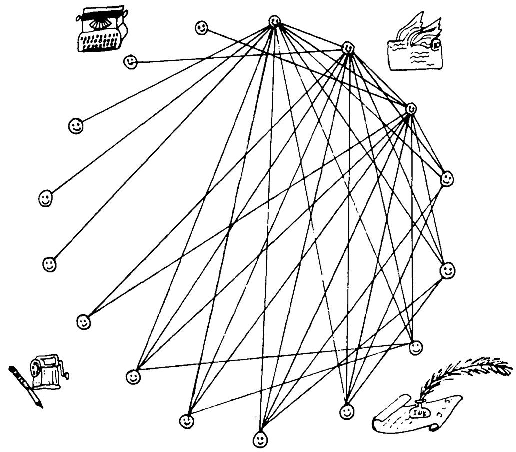


FIGURE 9

Replacing dots by 1's and blanks by 0's we obtain row 2 and column 2 of the 13×13 square submatrix we seek to construct (FIGURE 8). Striking out row 2 and column 2 of the configuration in FIGURE 7 yields a smaller Ferrers diagram on which we wish to repeat the procedure. For this case (since a new column of dots may have been created in adjusting the Ferrers diagram), it is not immediately obvious from the construction that the derived Ferrers diagram will satisfy all the conditions of the theorem. If (but only if) $d_{h+1} = h$ (in our case, $d_3 = 2$) and $k_h \geqslant h + d_h$, we get the new row sum $d_h^* = d_{h+1} = h$, and we must check that $\Sigma_1^h d_i^* \leqslant \Sigma_1^h (k_i^* - 1)$, since no corresponding inequality was included in the original conditions. But this is immediate, since

$$\sum_1^{h-1} d_i^* \leqslant \sum_1^{h-1} (k_i^* - 1).$$

Adding h to both sides, we get

$$\begin{aligned} \sum_1^{h} d_i^* &\leqslant \sum_1^{h-1} (k_i^* - 1) + h \\ &\leqslant \sum_1^{h-1} (k_i^* - 1) + k_h - d_h \\ &\leqslant \sum_1^{h} (k_i^* - 1). \end{aligned}$$

Thus the procedure can be repeated on the derived Ferrers diagram, and continued repetition produces the desired adjacency matrix. The completed adjacency matrix for our illustration and the pen pal plan are shown in FIGURE 9. Our example has illustrated all the difficulties that might arise given any partition satisfying the conditions of the theorem; a similar procedure would always give a graph having the required degree sequence.

The author is grateful to the editor for revisions which greatly improved and clarified this exposition.

References

[1] Frank Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969, pp. 57–63.

[2] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, NY, 1938, pp. 271–294.

[3] John Riordan, An Introduction to Combinatorial Analysis, Wiley, NY, 1958, pp. 107–117.

[4] Dorothy Wolfe, Graphical partitions of integers, submitted to Discrete Math.

The Probability of Election Reversal

JAMES E. WARD III
Bowdoin College
Brunswick, ME 04011

In a recent two-candidate Town Council election in Brunswick, Maine, the winner received 2,390 votes and the loser 2,383. After the vote had been announced, it was discovered that 16 voters had been allowed to vote at polling places where they were not officially registered, and those votes were declared invalid on the grounds that the voters had not been appropriately certified as being qualified to vote as required by state law. There was no evidence of fraud and no indication of how the invalid ballots were cast. The loser asked for a new election on the grounds that 12 or more of the 16 invalid votes *could* have been cast for the winner, enough to reverse the election results.

In this situation, the ballot box can be viewed as an urn in which there are a few more white balls (votes cast for the winner) than black balls (votes cast for the loser). A certain number of balls (the invalid votes) are withdrawn at random from the urn. What is the probability of a *reversal*, i.e., that, after the withdrawal, the number of black balls in the urn will exceed the number of white balls?

Of course the assumption that the balls are withdrawn at random is critical. The assumption that each ball has the same probability of being withdrawn is the assumption that each voter in the election has the same probability of casting an invalid ballot. If there is evidence of election irregularities or fraud, or if there is reason to assume that other factors, such as the time of day the invalid votes were cast or the location of the polling place, made it more likely that the invalid ballots were cast for one of the candidates, this assumption may not be tenable.

The probability of reversal is simply the number of withdrawal combinations which cause a reversal divided by the total number of possible withdrawal combinations. In most close elections, there will be too many of these combinations to be readily countable, so it is desirable to be able to approximate the probability of reversal. Because the arguments about election results occur in public and legal arenas, not in mathematical journals, a simple approximation is preferable.

An approximation due to Finkelstein and Robbins [2] is made as follows: suppose the urn contains w white balls and b black balls with $w > b$. Of the total $t = w + b$ balls, m are withdrawn at random. Let x denote the number of these m balls which are white. There will be a reversal if, after the withdrawal, there are at least as many black balls as white balls remaining in the urn; i.e., if $w - x \leqslant b - (m - x)$, or

$$x \geqslant \frac{w - b + m}{2}. \tag{1}$$

When m balls are withdrawn without replacement from $t = w + b$ balls, the number x of white balls withdrawn is a random variable with a hypergeometric distribution. The mean and variance of such a random variable are $E(x) = mw/t$ and $\operatorname{Var}(x) = mwb(t - m)/t^2(t - 1)$ [1]. Thus the standardized random variable $y = (x - E(x))/\sqrt{\operatorname{Var}(x)}$, having mean 0 and variance 1, is approximately

$$y = \frac{tx - mw}{\sqrt{mwb\left(\frac{t - m}{t}\right)}}. \tag{2}$$

(For large t, little accuracy is sacrificed by equating $t - 1$ and t.) Inequality (1) yields the following condition on the numerator of y:

$$tx - mw \geqslant \frac{t(w-b+m)-2mw}{2}. \tag{3}$$

Since $t = w + b$, the right-hand side of (3) is $(t-m)(w-b)/2$, so the condition (1) for reversal in terms of y is

$$y \geqslant \frac{(w-b)\sqrt{t(t-m)}}{2\sqrt{mwb}}. \tag{4}$$

Now $(w-b)^2 \geqslant 0$, so the equation $t = w + b$ implies that $t^2/4 \leqslant (w^2 + b^2)/2$. It also implies that $wb = (t^2/2) - (w^2 + b^2)/2$. Thus $wb \leqslant t^2/4$, with approximate equality when w and b are nearly equal. Hence (4) may be approximated by

$$y \geqslant (w-b)\sqrt{\frac{t-m}{tm}}. \tag{5}$$

That is, as y is approximately normally distributed, *the probability of a reversal is approximately equal to the probability that a standard normal random variable* y *will exceed the constant on the right-hand side of* (5).

Thus, to calculate the probability of an election reversal, calculate

$$z = d\sqrt{\frac{t-m}{tm}} \tag{6}$$

where d is the winner's majority (the vote difference), t is the total number of votes and m is the number of invalid votes. Then refer to a standard normal distribution table to find the probability of exceeding the value of z. The following part of that table gives a range of the probabilities:

Value of z	Probability of Reversal
.01	.50
.5	.31
1.0	.16
1.5	.07
2.0	.02
2.5	.006
3.0	.001

In the Maine election, $d = 7$, $t = 4{,}773$ and $m = 16$, so $z = 1.75$. From the table, the probability of reversal is .04. The fact that it was only 1 in 25 came as a disappointing surprise to the loser.

When t is large relative to m, as it is in the Maine election, a further simplification can be made, one which allows us to ignore t altogether. For then $\sqrt{(t-m)/t} \approx 1$ so that (6) can be replaced by

$$z = \frac{d}{\sqrt{m}}. \tag{7}$$

In the Maine election, (7) and (6) yield the same value (1.75) of z to two decimal places.

A solution to the original urn problem may also be obtained by assuming that the number of white balls withdrawn is a binomially distributed random variable X with parameter $p = .5$. That is, $X \sim b(m; .5)$. Calculations similar to those above for the standardized random variable z yield that the probability of reversal is

$$P\left[z \geqslant \frac{(m+d)/2-(m/2)}{\sqrt{m(.5)(.5)}}\right] = P\left[z \geqslant \frac{d}{\sqrt{m}}\right]$$

where m is the number of invalid ballots and d is the winner's majority. This is an alternative method of obtaining (7).

Equation (7) makes it clear that the probability of reversal depends primarily on the relative sizes of the winner's majority d and the number m of invalid votes. The total vote t is much less important. Moreover, it is clear that simple rules of thumb involving the relative sizes of d and m are hopelessly inaccurate (one example: call a new election if the number of invalid votes is twice the winner's majority). For instance, if the winner's majority in the Maine election had been 100 votes, it would have taken 2500 invalid ballots, twenty-five times the majority, to create even a .02 probability of reversal. It is apparent that m must increase much faster than d to maintain a constant probability.

If a legal body wants to determine a rule to apply to all election challenges, it must first decide how large the probability of reversal must be to warrant a new election. If that probability is determined to be .05, say, then (7) and the standard normal distribution table (which gives $P(z \geqslant .05) = 1.645$) yield that d and m must be such that $d/\sqrt{m} \leqslant 1.645$, or $d^2 \leqslant 2.7m$, to order a new election. If a .25 probability of reversal is required, then d and m must be such that $d^2 \leqslant .45m$ (as $P(z \geqslant .25) = .67$). In the Maine election with 16 invalid ballots, a new election would be ordered if the winner's margin were 6 or less under the .05 rule and 2 or less under the .25 rule. *In general, if a probability p of reversal is required to warrant a new election, then a new election should be ordered if $d^2 \leqslant \alpha^2 m$ where $P(z \geqslant p) = \alpha$ is determined from a standard normal distribution table.*

The probability of reversal has been used by legal bodies in reaching decisions on election challenges. In the Brunswick, Maine, election, the Town Council (the appellate body in this case) voted not to order a new election, citing as one of its reasons the small probability of reversal as determined by the chairman of the mathematics department at the local college.

However, courts and commissions are not always interested in or persuaded by probability arguments. Another Maine case will make the point. In a 1976 Maine House of Representatives District 89 election, the Republican defeated the Democrat by 133 votes out of 2,253 cast. After the result was announced, 208 of the votes were determined to be invalid and the Democrat asked the Maine Commission on Governmental Ethics and Election Practices to order a new election. The same mathematics chairman determined the probability of reversal to be infinitesimal ($z = 8.8$) and so testified to the Commission in a sworn affidavit. However, the Commissioners voted 2:1 to order a new election. The Commission chairman chose to dismiss the probability argument entirely on the grounds that

> Acceptance of a purely mathematical approach for predicting voter activity ignores myriad human variables generally perceived to be an inherent and important part of politics and generally perceived as working in no scientifically calculable manner. [**3**]

The other Commissioner voting for a new election, by profession a political science professor, was more logical and more interesting. In a concurring opinion, he wrote

> Our over-riding concern is reflected in the statement in the legislation which created the Commission: 'It is essential under the American system of representative government that the people have faith and confidence in the integrity of the elective process...' ... The voters could continue to believe that the election outcome might have been different. 'Common sense' indicates that 208 (invalid) votes could affect the outcome of a contest involving a margin of only 133 votes. The lay person does greatly overestimate the probabilities of reversal. As long as this occurs, however erroneously, faith and confidence in the accuracy of the outcome is undermined. ...Therefore the decision comes down to a matter of balancing competing weighty considerations. On the one hand, there (is) the extreme improbability that the election would have been reversed.... On the other hand, there are more subjective attempts somehow to restore faith and confidence in the electoral process... [**3**]

So a new election was held. And the second time the result was reversed and the Democrat won.

References

[1] M. H. DeGroot, Probability and Statistics, Addison-Wesley, Menlo Park, CA, 1975.
[2] M. O. Finkelstein and H. E. Robbins, Mathematical probability in election challenges, Columbia Law Review, 73, 24, pp. 241–248.
[3] Maine Commission on Governmental Ethics and Election Practices, Contested Ballot Appeal 76-CB-7.

Solving Exponential Diophantine Equations

LEO J. ALEX
SUNY, College at Oneonta
Oneonta, NY 13820

In the standard undergraduate introductory course in number theory, the notion of the order of one positive integer modulo another is introduced. This notion is then used to motivate the idea of a primitive root, and the theorem detailing which positive integers have primitive roots is given. Then the notion of order is dropped and all too soon forgotten.

In this note an application of the notion of order to the solution of simple exponential Diophantine equations is given. This application nicely reinforces the ideas of congruences and order for the student.

If a and m are positive integers which are relatively prime, then the **order of a mod m**, denoted $\mathrm{ord}_m a$, is the smallest positive integer t such that $a^t \equiv 1 \bmod m$. We will need the following proposition, which follows immediately from the division algorithm and the Euler-Fermat Theorem and which can be found in any standard modern algebra or number theory text.

PROPOSITION. *Let a and m be positive integers which are relatively prime, and let $t = \mathrm{ord}_m a$. Then*

(1) *t divides $\phi(m)$, the number of positive integers relatively prime to m, and*
(2) *if k is any positive integer such that $a^k \equiv 1 \bmod m$, then t divides k.*

Consider the Diophantine equation

$$1 + x = y \tag{E}$$

where we seek positive integer solutions x, y subject to the condition that the set S of all possible prime divisors of the product xy is specified. In this situation, the notion of order (especially part (2) of the Proposition), elementary properties of congruence, and a little ingenuity give neat solutions to equation (E).

EXAMPLE 1. Solve equation (E) when $S = \{2,3\}$.

Solution. Since x and y are relatively prime, it follows that the equation must be one of the forms:

$$\text{(i)} \qquad 1 + 3^a = 2^b$$

$$\text{(ii)} \qquad 1 + 2^a = 3^b,$$

where a and b are nonnegative integers. In case (i) if $a > 1$, then $2^b \equiv 1 \bmod 9$. But $\mathrm{ord}_9 2 = 6$. Thus by part (2) of the Proposition, 6 divides b. But $\mathrm{ord}_7 2$ is 3. Thus $2^b \equiv 2^{6r} \equiv (2^3)^{2r} \equiv 1 \bmod 7$. Thus 7 divides 3^a, a contradiction. Hence $a \leqslant 1$ and the solutions to equation (E) in this case are $(x, y) = (1,2)$ and $(x, y) = (3,4)$. Similarly in case (ii), if $a > 3$, then $3^b \equiv 1 \bmod 16$. Since $\mathrm{ord}_{16} 3 = 4$, it follows that $b \equiv 0 \bmod 4$. But then since $\mathrm{ord}_5 3 = 4$, we obtain the contradiction that 5 divides 2^a. Thus in this case $a \leqslant 3$, and the solutions of ordered pairs $(x, y) = (2,3)$ and $(8,9)$ are determined.

References

[1] M. H. DeGroot, Probability and Statistics, Addison-Wesley, Menlo Park, CA, 1975.
[2] M. O. Finkelstein and H. E. Robbins, Mathematical probability in election challenges, Columbia Law Review, 73, 24, pp. 241–248.
[3] Maine Commission on Governmental Ethics and Election Practices, Contested Ballot Appeal 76-CB-7.

Solving Exponential Diophantine Equations

LEO J. ALEX
SUNY, College at Oneonta
Oneonta, NY 13820

In the standard undergraduate introductory course in number theory, the notion of the order of one positive integer modulo another is introduced. This notion is then used to motivate the idea of a primitive root, and the theorem detailing which positive integers have primitive roots is given. Then the notion of order is dropped and all too soon forgotten.

In this note an application of the notion of order to the solution of simple exponential Diophantine equations is given. This application nicely reinforces the ideas of congruences and order for the student.

If a and m are positive integers which are relatively prime, then the **order of a mod m**, denoted $\text{ord}_m a$, is the smallest positive integer t such that $a^t \equiv 1 \bmod m$. We will need the following proposition, which follows immediately from the division algorithm and the Euler-Fermat Theorem and which can be found in any standard modern algebra or number theory text.

PROPOSITION. *Let a and m be positive integers which are relatively prime, and let $t = \text{ord}_m a$. Then*

(1) *t divides $\phi(m)$, the number of positive integers relatively prime to m, and*
(2) *if k is any positive integer such that $a^k \equiv 1 \bmod m$, then t divides k.*

Consider the Diophantine equation

$$1 + x = y \tag{E}$$

where we seek positive integer solutions x, y subject to the condition that the set S of all possible prime divisors of the product xy is specified. In this situation, the notion of order (especially part (2) of the Proposition), elementary properties of congruence, and a little ingenuity give neat solutions to equation (E).

EXAMPLE 1. Solve equation (E) when $S = \{2,3\}$.

Solution. Since x and y are relatively prime, it follows that the equation must be one of the forms:

$$\text{(i)} \qquad 1 + 3^a = 2^b$$

$$\text{(ii)} \qquad 1 + 2^a = 3^b,$$

where a and b are nonnegative integers. In case (i) if $a > 1$, then $2^b \equiv 1 \bmod 9$. But $\text{ord}_9 2 = 6$. Thus by part (2) of the Proposition, 6 divides b. But $\text{ord}_7 2$ is 3. Thus $2^b \equiv 2^{6r} \equiv (2^3)^{2r} \equiv 1 \bmod 7$. Thus 7 divides 3^a, a contradiction. Hence $a \leqslant 1$ and the solutions to equation (E) in this case are $(x, y) = (1,2)$ and $(x, y) = (3,4)$. Similarly in case (ii), if $a > 3$, then $3^b \equiv 1 \bmod 16$. Since $\text{ord}_{16} 3 = 4$, it follows that $b \equiv 0 \bmod 4$. But then since $\text{ord}_5 3 = 4$, we obtain the contradiction that 5 divides 2^a. Thus in this case $a \leqslant 3$, and the solutions of ordered pairs $(x, y) = (2,3)$ and $(8,9)$ are determined.

EXAMPLE 2. Solve $1+3^a=2^b5^c$.

Solution. If a is odd, $2^b5^c \equiv (1+3) \bmod 8$, so $b=2$. Also, if a is odd, $1+3^a \not\equiv 0 \bmod 5$, so $c=0$. Then $1+3^a=4$ gives the only solution with a odd, which is $(a,b,c)=(1,2,0)$. If a is even, $2^b5^c \equiv (1+1) \bmod 4$, so $b=1$. If $c>1$, then $1+3^a \equiv 0 \bmod 25$ which yields $a \equiv 10 \bmod 20$. Thus $1+3^a=1+(3^{10})^{1+2t}$. But the prime 1181 divides $3^{10}+1$, so $1+3^a \equiv 1+(-1)^{1+2t} \equiv 0 \bmod 1181$ which gives a contradiction. Thus $c \leqslant 1$ and the only solution with a even is found to be $(a,b,c)=(2,1,1)$.

The method will always solve equations of the form $1+x=y$ where the primes dividing xy are specified. Ingenuity is sometimes needed, the amount varying with the equation. The necessary moduli (as 1181 above) can be quite large in comparison with the primes in xy. For example, to solve $1+5^a7^b73^c=2^d$, one can follow this sequence of steps: (i) if $a>0$, consider the equation mod 5 and then mod 3 to get a contradiction, (ii) if $b>1$, consider the equation mod 49 and then mod 127 to get a contradiction, (iii) if $c>1$, consider the equation mod 5329 ($5329=73^2$) to get $2^d \equiv 1 \bmod 5329$ and use the facts that $\operatorname{ord}_{5329}2=657$ and 439 divides $2^{73}-1$ to get $2^d \equiv 1 \bmod 439$, which gives a contradiction. So, the only solutions to the equation are $1+1=2$, $1+7=8$, and $1+511=512$.

For the instructor or student who wishes to penetrate further into the realm of exponential Diophantine equations, there exists a modest amount of literature. In **[5]** Lehmer notes that interest in pairs (x,y) satisfying equation (E) dates back at least to the 18th century and seems to have been awakened by their usefulness in calculating logarithms of integers to great accuracy. Also such pairs can be used to find particular solutions of Diophantine equations of the form $Ar^n-Bt^m=1$. For example, $r^2-14t^3=1$ has the solution $(r,t)=(55,6)$ because the pair $(x,y)=(3024, 3025)$ is a solution to equation (E). Størmer **[7]** showed in 1897 that equation (E) has only a finite number of solutions when S is specified, and he found all solutions when $S=\{2,3,5,7\}$. His method involved solving related Fermat-Pell equations. Lehmer **[5]** extended this work and listed all solutions to the equation with S consisting of all primes $p \leqslant 41$. Alex **[1]** used the methods of Examples 1 and 2 above to find all solutions to equation (E) and the related equation $x+y=z$ where the set S of possible primes dividing xyz is $S=\{2,3,5,7\}$. Alex also gives connections between these equations and finite group character theory. Cassels **[2]** showed that the related equation

$$p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}-p_{k+1}^{e_{k+1}}p_{k+2}^{e_{k+2}}\cdots p_r^{e_r}=C$$

has only a finite number of solutions which can be found in a finite number of steps. Halsey and Hewitt **[4]** and Silver **[6]** showed that the solutions to equation (E) with $S=\{2,3,5\}$ correspond to the preferred superparticular ratios (intervals) in music. For example, the ratio $y/x=2/1$ corresponds to the octave, and $y/x=6/5$ to the major third. Finally, the methods of Examples 1 and 2 can be used to solve more general exponential equations. The reader may, for example, try the equation $1+2^a=3^b+2^c$ as an exercise. Solutions to two equations like this appear in **[3]**.

References

[1] L. J. Alex, Diophantine equations related to finite groups, Comm. Algebra, 4 (1) (1976) 77–100.

[2] J. W. S. Cassels, On a class of exponential equations, Ark. Mat., 4 (1961) 231–233.

[3] Lorraine L. Foster, Equal sums of powers of primes; solution to Proposal E2749, Amer. Math. Monthly, 87 (1980) 138.

[4] G. D. Halsey and Edwin Hewitt, More on the superparticular ratios in music, Amer. Math. Monthly, 79 (1972) 1096–1100.

[5] D. H. Lehmer, On a problem of Størmer, Illinois J. Math., 8 (1964) 57–79.

[6] A. L. Leigh Silver, Musimatics or the nun's fiddle, Amer. Math. Monthly, 78 (1971) 351–357.

[7] Carl Størmer, Quelques théorèmes sur l'équation de Pell $x^2-Dy^2=\pm 1$ et leurs applications, Skrifter Videnskalsselskabet (Christiana) I, Mat.-Naturv. kl, 1897, no. 2 (48 pp).

Average Lengths of Chords in a Square

PHILIP W. KUCHEL
University of Sydney
New South Wales, Australia, 2006

RODNEY J. VAUGHAN
University of Newcastle
New South Wales, Australia, 2308

What is the average length of a chord in a square? This problem arose originally from the somewhat whimsical desire to determine the length of the ink path in a hand-written manuscript. If the written letters are placed under a fine transparent grid of uniform squares (e.g., FIGURE 1 shows one such magnified letter), the length of a word can be determined by counting the number of squares traversed by the curve and then simply multiplying this value by the average length of the line in the square. The method clearly is a first approximation to the true line integral of the complicated curve that constitutes the written script; it assumes that the curve can be represented as a series of linear segments of varying length and that the centre of the path is unambiguously defined. The approximation clearly improves with a reduction in the size of the elements of the grid.

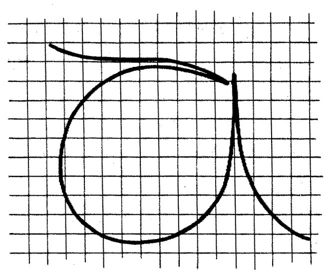


FIGURE 1

The procedure should be applicable to modern electronic digitizing pads where the path taken by the reading head is recorded as a series of (x, y) coordinates which are disposed on a fine square grid.

In order to answer the opening question, we need to first define what we mean by "average length of a chord." A more precise statement of our problem is: *determine the average length of randomly oriented straight lines cut off by a square in the Euclidean plane*. The average length will clearly depend on *how* the randomness of the orientation of the family of chords is defined, and several cases will be considered here.

This type of problem is encompassed by the realm of what is known as statistical geometry or geometric probability. One of the fundamental classes of problems addressed in this subject is the determination of the lengths of lines in geometric figures where the ends of the lines are not specified absolutely but are distributed according to probability distribution functions; it follows, of course, that analogous questions relating to dimensions greater than 2 can be addressed.

Statistical geometry has been studied over the past century beginning with the work of Crofton [**2**] and a more recent example of its application has been the determination of the average distance between two random points in a circle [**1**], [**4**]. Another two examples have been the determination of the average distance between two random points in adjacent circles [**3**] and between two adjacent squares [**9**]. The most recent treatment of the general field appears in Solomon [**7**] whilst the book of Kendall and Moran [**5**] remains invaluable.

Our discussion considers the potentially more complex problem of evaluating the chord lengths determined by random points and lines in squares, but in the interests of brevity we will resist the obvious temptation to consider analogous problems dealing with circles. Such excursions may tempt the reader!

There are several ways of defining a random line in a square of side l and each yields a different value for the mean relative (to l) length of the line. The physical basis for the choice of one particular solution over and above another will depend on how the estimates of individual line lengths are recorded in the first place. We will therefore consider each of the cases relevant to this problem.

CASE I. The line describing a closed, basically circular, letter (such as the letter *a* in FIGURE 1) can be thought of as taking on all possible orientations so that the segments cut off by a fine square grid on which it is drawn can be viewed as part of the set of randomly orientated lines in the plane. This situation leads to the following abstract definition of the problem:

> *In the classical concept of geometrical probability, a figure (in this case a square of side l) is considered to be lying in a field of random lines. These lines have a continuum of orientations which are equally probable and the density of lines for each direction is the same.*

In this case the average length of the chord AB (denoted by S_1, for the sake of consistency with later cases) formed by the random lines is $\pi/4$, or approximately $0.7854l$ [7]. However, discrepancies clearly arise in applying this case to those script characters (such as the letter t) with sharp curvatures and biased orientations of their various segments.

A more realistic solution may be forthcoming by considering the use of an electronic digitizing pad connected to a computer. Some types of these devices employ a fine grid of parallel wires running at right angles to each other, but of course, electrically isolated. The reading head passes over the grid and its location is recorded, by the use of appropriate electronics, by the effect the head has on a current in the wires immediately beneath it. From the point of view of solutions to the problem presented here, this situation is equivalent to starting at a random point on one side of a grid element (square) and proceeding to another random point on any side of the square. This leads us to our second general case.

CASE II.

> *Two independent points A and B move anywhere on the four sides of the square of side l, and cut off the chord AB. The position of one point does not restrict that of the other.*

For pedagogical reasons it is useful to consider first some subcases of this general case and the first is as follows.

CASE II(i).

> *The two points A and B are on opposite sides of the square, each having equal probability of being at any point on that side.*

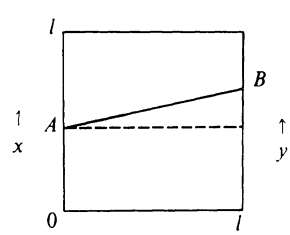


FIGURE 2

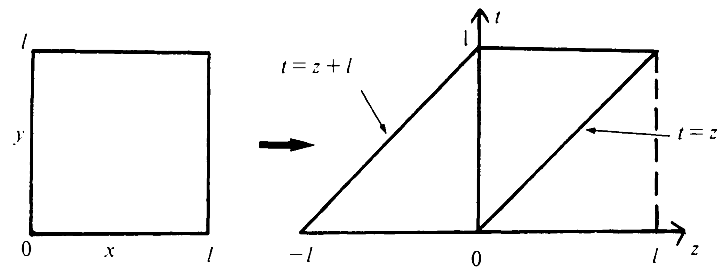


FIGURE 3

The configuration of the square of side l and points $A(x)$ and $B(y)$ is shown in FIGURE 2. The probabilities that A and B lie in the domains $[x, x+dx]$ and $[y, y+dy]$ are dx/l and dy/l respectively. Therefore, the probability that A and B simultaneously lie in the intervals is $dx\,dy/l^2$ and

$$\text{the length of } AB = \left(l^2 + (y-x)^2\right)^{\frac{1}{2}}. \tag{1}$$

Hence, the average length of AB (denoted by S_2 in the following equation) is given by the integral (sum) of all these lengths weighted with respect to the above probability element:

$$S_2 = \frac{\int_0^l \int_0^l \left(l^2 + (y-x)^2\right)^{\frac{1}{2}} dx\,dy}{l^2}. \tag{2}$$

In order to evaluate the integral, the following transformation is useful:

$$z = x - y$$
$$t = x.$$

The effect of this transform is illustrated in FIGURE 3.

It follows from standard calculus that

$$S_2 = \frac{1}{l^2}\int_0^l \int_z^l (l^2 + z^2)^{\frac{1}{2}} dt\,dz + \frac{1}{l^2}\int_{-l}^0 \int_0^{z+l} (l^2 + z^2)^{\frac{1}{2}} dt\,dz$$
$$= \frac{2}{l^2}\int_0^l (l-z)(l^2+z^2)^{\frac{1}{2}} dz.$$

This is a standard integral ([**8**], 14.189 and 14.190) so that

$$S_2 = \frac{2}{l^2}\left\{\frac{l}{2}\left[z(z^2+l^2)^{\frac{1}{2}} + l^2\ln\left(z + (z^2+l^2)^{\frac{1}{2}}\right)\right]\Bigg|_0^l - \frac{1}{3}(z^2+l^2)^{\frac{3}{2}}\Bigg|_0^l\right\}$$
$$= \frac{2}{l^2}\left\{\frac{l}{2}\left[\sqrt{2}\,l^2 + l^2\ln(1+\sqrt{2})\right] - \frac{l^3}{3}(2\sqrt{2}-1)\right\}$$
$$\doteqdot 1.0766l. \tag{3}$$

A second subcase is as follows.

CASE II(ii).

The two points A and B are on adjacent sides of a square each having equal probability of being at any point on that side.

This is the same as the previous situation except that the distance from A to B is $(x^2+y^2)^{\frac{1}{2}}$,

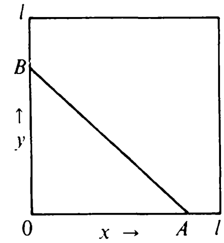


FIGURE 4

and it can be seen that the average value of the length of AB (denoted S_3), as in FIGURE 4, is given by

$$S_3 = \frac{1}{l^2} \int_0^l \int_0^l (x^2 + y^2)^{\frac{1}{2}} dx\, dy. \tag{4}$$

We transform to polar coordinates in the usual manner:

$$r = (x^2 + y^2)^{\frac{1}{2}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Now the region of integration (FIGURE 5) is more complex, but the resulting integrals are more readily solved.

With this transformation, (4) becomes

$$S_3 = \frac{1}{l^2}\left\{\int_0^{\pi/4}\int_0^{l\sec\theta} r^2\, dr\, d\theta + \int_{\pi/4}^{\pi/2}\int_0^{l\csc\theta} r^2\, dr\, d\theta\right\}$$

$$= \frac{2}{l^2}\int_0^{\pi/4} \frac{r^3}{3}\bigg|_0^{l\sec\theta} d\theta$$

$$= \frac{2}{3l^2}\int_0^{\pi/4} l^3 \sec^3\theta\, d\theta.$$

Again this is a standard integral ([**8**], 14.451) and

$$S_3 = \frac{l}{3}\left[\sqrt{2} + \ln(1+\sqrt{2})\right] \doteqdot 0.7652l. \tag{5}$$

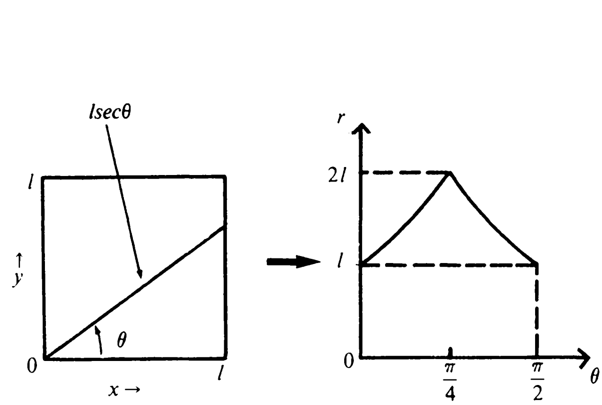


FIGURE 5

Finally, the simplest subcase must be solved in order to arrive at the complete solution of Case II.

CASE II(iii).

The two points A and B are on the same side of a square (i.e., on a line) with equal probability of being at any point on the side.

The solution to this problem is well known [5], and the average length of AB (given here as S_4) is

$$S_4 = l/3 \doteqdot 0.3333l. \tag{6}$$

We are now in a position to find the average length of AB as defined in Case II because we have considered all possible subcases. In general the probability that the two random points A and B lie on opposite sides of a square (as in Case II(i)) is $\frac{1}{4}$, since only 1 side out of the total of 4 is defined as being opposite to a given side. Using analogous reasoning, we see that the probability that A and B lie on adjacent sides of the square (as in Case II(ii)) is $\frac{1}{2}$. Finally, the probability that A and B lie on the same side is $\frac{1}{4}$. Therefore the average length of AB (defined as S_5 for Case II) is given by the weighted sum

$$S_5 = \frac{1}{4}S_2 + \frac{1}{2}S_3 + \frac{1}{4}S_4 \doteqdot 0.7351l. \tag{7}$$

When drawing a curve on a digitizing pad (an electronic means whereby x and y coordinates of the reading head are updated by the computer), the computer will register only changes in the x and/or y coordinates, i.e., when the reading head traverses another wire. The path length taken by the head is then obtained by summing the small linear segments defined by the set of x, y coordinates. This technique of recording coordinates (one side of a square grid element corresponds to a pair of x, y coordinates) has the effect that if the head passes back over the same wire (side of square grid element) without first traversing another wire, then the path taken is not recorded. In other words, the head will not record movements which enter (point A in our cases) and leave (point B) via the *same* side. This suggests that the complete solution to the problem given in Case II is not applicable to the digitizing pad system described here. It follows that the appropriate average length of chord (S_6) in the square in this situation is given by

$$S_6 = \frac{1}{3}S_2 + \frac{2}{3}S_3 \doteqdot 0.8690l, \tag{8}$$

where S_2 and S_3 are defined in (2) and (4).

Another means of defining the random orientation of segments of the ink path taken by a written script as it traverses a square grid is to suggest that the writing of a character (randomly chosen) begins on the side of a square at a random point A. The line then proceeds with random orientation to the side of an adjacent square (to point B, regarded as new point A) whereupon it in turn proceeds to its particular adjacent square with a new orientation (defined by whichever character is being drawn). Viewed in this way the appropriate solution is obtained from the following case defined in abstract terms as,

CASE III.

The single point A is allowed to move with an equal probability of being at any point on the side of the given square, and a straight line through A intersecting the square at B has an angular orientation (θ) the probability distribution of which is uniform; i.e., the probability of the angle of the line lying in the domain $[\theta, \theta + d\theta]$ is equal to $d\theta$.

Reference to FIGURE 6 will show that there are essentially two distinct expressions for the length of AB, depending on whether the "random" line cuts adjacent or opposite sides. Furthermore the angular limits which define the domain of θ over which the expression holds are readily obtained:

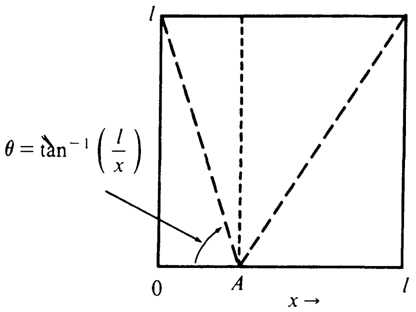


FIGURE 6

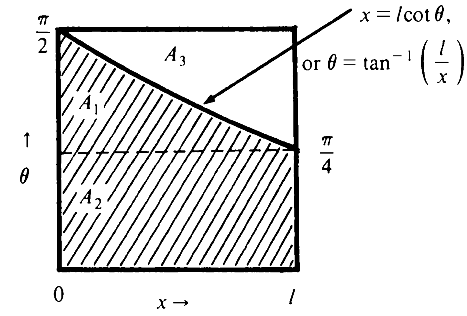


FIGURE 7

$$\text{length of } AB = \begin{cases} x\sec\theta & 0 \leqslant \theta \leqslant \tan^{-1}\left(\frac{l}{x}\right) \\ l\csc\theta & \tan^{-1}\left(\frac{l}{x}\right) \leqslant \theta \leqslant \frac{\pi}{2}. \end{cases} \tag{9}$$

Consideration of the symmetry in FIGURE 6 indicates that A can be envisaged as confined to one side of the square and θ restricted to the domain $[0, \pi/2]$ with no loss of generality.

As the probability element in this case is $d\theta\, dx/(\pi/2)$, the average length of AB (denoted as S_7) is given by

$$S_7 = \frac{2}{\pi l}\left\{\int_0^l \int_0^{\tan^{-1}\left(\frac{l}{x}\right)} x\sec\theta\, d\theta\, dx + \int_0^l \int_{\tan^{-1}\left(\frac{l}{x}\right)}^{\pi/2} l\csc\theta\, d\theta\, dx\right\}. \tag{10}$$

To evaluate S_7, change the order of integration; when $x = 0$, $\theta = \pi/2$, when $x = l$, $\theta = \pi/4$.

The three integrals which occur below in the expression for S_7 relate to the three areas A_1, A_2, and A_3 in FIGURE 7.

$$S_7 = \frac{2}{\pi l}\left\{\int_{\pi/4}^{\pi/2}\int_0^{l\cot\theta} x\sec\theta\, dx\, d\theta + \int_0^{\pi/4}\int_0^l x\sec\theta\, dx\, d\theta + \int_{\pi/4}^{\pi/2}\int_{l\cot\theta}^{l} l\csc\theta\, dx\, d\theta\right\}$$

$$= \frac{2}{\pi l}\left(\frac{l^2}{2}\right)\left\{\int_{\pi/4}^{\pi/2}\csc\theta\cot\theta\, d\theta + \int_0^{\pi/4}\sec\theta\, d\theta + 2\int_{\pi/4}^{\pi/2}\csc\theta\, d\theta - 2\int_{\pi/4}^{\pi/2}\csc\theta\cot\theta\, d\theta\right\}.$$

These integrals are standard forms ([**8**]; 14.345, 14.375, 14.427) so that

$$S_7 = \frac{l}{\pi}\left\{2\ln(\csc\theta - \cot\theta)\Big|_{\pi/4}^{\pi/2} + \ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} + \csc\theta\Big|_{\pi/4}^{\pi/2}\right\}$$

$$= \frac{l}{\pi}\left\{-2\ln(\sqrt{2} - 1) + \ln(\sqrt{2} + 1) + (1 - \sqrt{2})\right\}$$

$$\doteqdot 0.7098l. \tag{11}$$

Rather than specifying that the written letter commences on the side of a grid element (square), a less restrictive condition is that the pen starts at some random point inside a square (grid element) and the line then proceeds with random orientation through the side of the square to a new random point in an adjacent square. For any character, the next point could be envisaged to be anywhere along the path of the line within the adjacent square. If this argument holds, then the mean length of the lines in the squares must be evaluated for the following abstract situation.

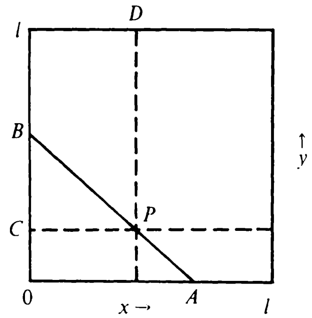


FIGURE 8

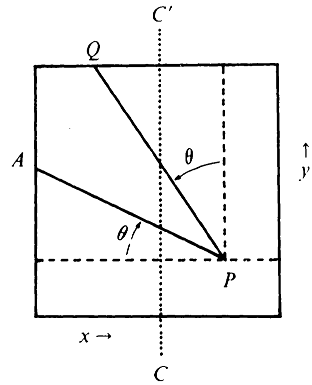


FIGURE 9

CASE IV.

The point P is allowed to move in and on the square to any location with uniform probability, and a straight line from a uniform probability angle distribution passes through it to intersect the square at A and B.

Consider the arrangement depicted in FIGURE 8.

The chord APB is comprised of two lengths AP and PB. The section AP is contained in the rectangle of area $y \times l$ and the section PB in the rectangle of area $(l-y) \times l$. Therefore, in order to find the average length of the chord APB, with respect to the angle θ rotated about P, it is necessary to calculate the average length of AP and PB with respect to θ. To generate P at random, x and y are allowed to follow a uniform distribution. It follows that because of symmetry the average lengths of AP and PB (with respect to θ, x and y) are equal. Hence, the problem reduces to one of finding the length of AP and doubling the value found.

Consider the rectangle of the area $y \times l$ in FIGURE 8; then using symmetry as in Case III, the average length of AP with respect to θ and x can be obtained by restricting θ to the domain $[0, \pi/2]$. In this case,

$$\text{length of } AP = \begin{cases} x \sec\theta & 0 \leqslant \theta \leqslant \tan^{-1}(y/x) \\ y \csc\theta & \tan^{-1}(y/x) \leqslant \theta \leqslant \pi/2. \end{cases} \tag{12}$$

Therefore, using the above-mentioned features, and the observation that the probability element is $d\theta\, dx\, dy/(\pi l^2/2)$, it follows that the average chord length in this case is

$$S_8 = 2\left(\frac{2}{\pi l^2}\right)\left\{\int_0^l\int_0^l\int_0^{\tan^{-1}\left(\frac{y}{x}\right)} x\sec\theta\, d\theta\, dx\, dy + \int_0^l\int_0^l\int_{\tan^{-1}\left(\frac{y}{x}\right)}^{\pi/2} y\csc\theta\, d\theta\, dx\, dy\right\}. \tag{13}$$

Denote the two triple integrals in (13) by I_1 and I_2, respectively. By a piece of good fortune, the integrals I_1 and I_2 are equal. To see this, consider FIGURE 9. I_1 yields the average (with respect to θ, x and y) length of AP whilst I_2 yields QP. It can be seen that the average length will be equivalent if

(i) x and y are interchanged,

(ii) FIGURE 9 is rotated clockwise through an angle of $\pi/2$ about the centre of the square,

(iii) FIGURE 9 is reflected about CC'.

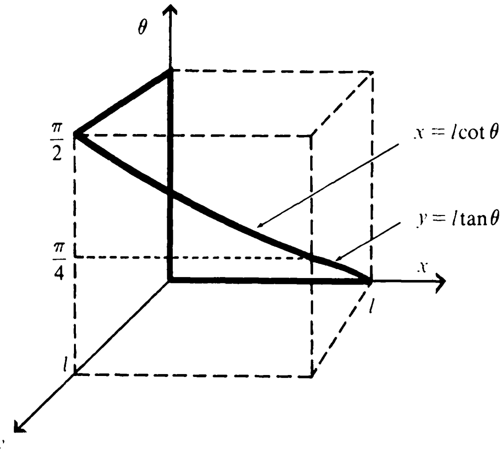


FIGURE 10

Now to evaluate I_1, consider FIGURE 10, where it is left to the reader to imagine the surface $\theta = \tan^{-1}(y/x)$ with the boundary as indicated by the heavy lines. The volume under this surface is the region over which the function $x\sec\theta$ must be integrated. However, it is much simpler if the surface is described explicitly in terms of x and θ, i.e., $y = x\tan\theta$. An easily evaluated integral results if the order of integration is first y, then x, then θ. For this, it is required that $x\tan\theta \leqslant y \leqslant l$ with x and θ in the shaded region of FIGURE 7. This area can be broken further into two subareas A_1 and A_2:

$$\begin{aligned} &A_1\colon 0 \leqslant x \leqslant l\cot\theta, \qquad \frac{\pi}{4} < \theta < \frac{\pi}{2} \\ &A_2\colon 0 \leqslant x \leqslant l, \qquad 0 < \theta \leqslant \frac{\pi}{4}. \end{aligned} \tag{14}$$

Hence the volume of integration is divided into two subvolumes:

$$\begin{aligned} &V_1\colon x\tan\theta \leqslant y \leqslant l, \qquad 0 \leqslant x \leqslant l\cot\theta, \frac{\pi}{4} < \theta < \frac{\pi}{2} \\ &V_2\colon x\tan\theta \leqslant y \leqslant l, \qquad 0 \leqslant x \leqslant l, 0 < \theta \leqslant \frac{\pi}{4}. \end{aligned} \tag{15}$$

Thus

$$\begin{aligned} I_1 &= \int_0^{\pi/4}\int_0^l\int_{x\tan\theta}^l x\sec\theta\,dy\,dx\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{l\cot\theta}\int_{x\tan\theta}^l x\sec\theta\,dy\,dx\,d\theta \\ &= \int_0^{\pi/4}\int_0^l x\sec\theta(l - x\tan\theta)\,dx\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{l\cot\theta} x\sec\theta(l - x\tan\theta)\,dx\,d\theta \\ &= \int_0^{\pi/4}\left[\frac{l^3}{2}\sec\theta - \frac{l^3}{3}\tan\theta\sec\theta\right]d\theta + \int_{\pi/4}^{\pi/2}\frac{l^3}{6}\cot\theta\csc\theta\,d\theta \\ &= l^3\left[\frac{1}{2}\ln(\sqrt{2}+1) - \frac{1}{6}(\sqrt{2}-1)\right] \\ &\doteq 0.3716 l^3. \end{aligned}$$

From this, and our earlier remarks about equation (13), we obtain

$$S_8 = \frac{8}{\pi l^2}(I_1) \doteq 0.9464 l. \tag{16}$$

Case	Conditions	Average Length of Chord
I	Random field of lines	$0.7854l$
II	Two random points on any side of square	$0.7351l$
	Two random points on perimeter of square but not on same side	$0.8690l$
II(i)	Random points on opposite sides of square	$1.0766l$
II(ii)	Random points on adjacent sides of square	$0.7652l$
II(iii)	Random points on same side of square	$0.3333l$
III	Single point on any side of square with a line of random uniform orientation through it	$0.7098l$
IV	Single point anywhere in or on square with line of random orientation through it	$0.9464l$

TABLE 1. **The average length of random chords in a square of side l.**

We have determined the average length of chords in a square of side l for four fundamentally different ways of defining a random distribution of lines in a square and the results are summarized in TABLE 1. There is a striking difference between the values obtained for the four main cases, which highlights the importance of always specifying what a "random line" really is. There does not appear to us to be any really persuasive argument for the general use of one or other of the solutions. However, with the advent of a sophisticated computer-based, character-generating algorithm, it is now possible to define numerals and letters as complicated polynomial functions [**6**]. The true line integral of these functions should be available and could therefore enable an evaluation of which of our solutions is the most appropriate.

Finally, from a purely mathematical point of view, despite the lack of a truly definitive answer to the practical problem introduced here, the search for a solution has led to the evaluation of some very interesting integrals!

References

[1] H. G. Apsimon, Mathematical Note 2754—A repeated integral, Math. Gaz., 42 (1958) 52.

[2] M. Crofton, "Probability," in Encyclopaedia Britannica, 9th ed., 1885, vol. 19, pp. 771–788.

[3] D. Fairthorne, The distance between random points in two circles, Biometrika, 51 (1964) 275–277.

[4] F. Garwood and J. C. Tanner, Note 2800 on Note 2754—A repeated integral, Math. Gaz., 42 (1958) 292–293.

[5] M. G. Kendall and P. A. P. Moran, Geometrical Probability, Griffin, London, 1963.

[6] D. E. Knuth, Art of Computer Programming—Sub-Series Numerical Algorithms, Addison-Wesley, Reading, MA, 1980, vol. 2, 2nd ed.

[7] H. Solomon, Geometric Probability, SIAM (1978).

[8] M. R. Spiegel, Mathematical Handbook of Formulas and Tables, Schaum's Outline Series, McGraw-Hill, NY, 1968.

[9] R. J. Vaughan, Solution of problem 75-12, SIAM Rev., 18 (1976) 500.

PROBLEMS

DAN EUSTICE, Editor
LEROY F. MEYERS, Associate Editor
The Ohio State University

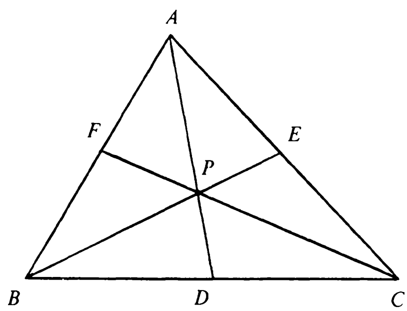
Proposals

To be considered for publication, solutions should be mailed before May 1, 1982.

1132. Let triangle ABC be given and labeled as shown in the figure.

(a) Show that if AD bisects angle A and $BD \cdot CE = DC \cdot BF$, then ABC is an isosceles triangle.

(b) Show that if AD, BE, and CF bisect angles A, B, and C respectively, and $BP \cdot FP = BF \cdot AP$, then ABC is a right triangle. [*Roger Izard, Dallas, Texas.*]



1133. Let f_{nk}, $k = 1, \ldots, n$, be n polynomials of degree n defined by

$$f_{nk}(x) = \frac{(-x)^k}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \left[\frac{(1-x)^n - 1}{x} \right].$$

Show the following properties of f_{nk}:

(a) For each x in $(0,1)$, $f_{n1}(x) > f_{n2}(x) > \cdots > f_{nn}(x)$.

(b) Each f_{nk} is strictly increasing on $[0,1]$ with $f_{nk}(0) = 0$ and $f_{nk}(1) = 1$.

(c) $\frac{1}{n} \sum_{k=1}^{n} f_{nk}(x) = x$.

(d) $\int_0^1 f_{nk}(x)\, dx = 1 - \frac{k}{n+1}$. [*G. Z. Chang, University of Utah.*]

ASSISTANT EDITORS: DON BONAR, *Denison University;* WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to L. F. Meyers, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

1134. Let $a \geq 0$ be fixed. Find

$$\sup\left\{\frac{\log s}{t^a} : 0 < s < t < 1 \text{ and } s\log s = t\log t\right\}.$$

[*Benjamin G. Klein, Davidson College.*]

1135. Let $f(z) = \sum_{n=1}^{\infty}(1-ab)(1-a^2b)\cdots(1-a^nb)z^n$, where $|a| < 1$ and $a^nb \neq 1$ for $n \geq 1$.

(a) Find the radius of convergence and show that f can be continued analytically to a meromorphic function f^* on $\mathbb{C}$, the complex plane.

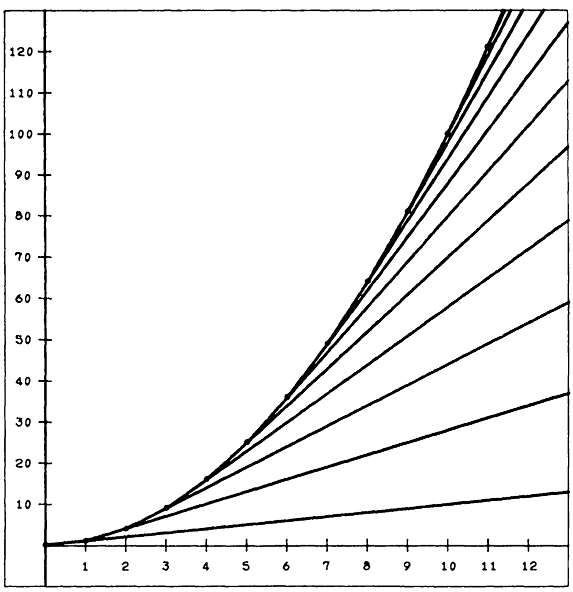
(b) Find the residues at the poles of f^* and exhibit a series representation for f^*. [*Paul Schweitzer, University of Rochester.*]

Solutions

1,000 = W. that a P. is W. **September 1980**

1101. Can the plane be tiled with infinitely many convex regions such that every straight line meets only a finite number of the regions? [*Daniel B. Shapiro, The Ohio State University.*]

Solution: (Without Words)



(Boundary lines are the y-axis, and for $i = 0, 1, 2, 3, \ldots,$ the half-line from (i, i^2) (initial point) passing through $(i+1, (i+1)^2)$.)

ST. OLAF PROBLEM SOLVING GROUP
St. Olaf College

Also solved by Richard Beigel, F. Cunningham, Jr., Raymond A. Maruca, J. L. Selfridge, and the proposer.

Bilinear Forms September 1980

1102. Let B and B' be bilinear forms on a vector space V over a field F. Suppose that for every x and y in V, $B(x,y)=0$ implies $B'(x,y)=0$. Prove that $B=cB'$ for some c in F. [*Daniel B. Shapiro, The Ohio State University.*]

Solution: As stated the result is false: let $B'(x,y)\equiv 0$ and let $B(x,y)$ be a nonzero bilinear form. The correct relationship is $B'=cB$. [Our mistake—*Editors.*]

If $B(u,v)=0$, then $B'(u,v)=cB(u,v)$ no matter what c is. Therefore, let $v_1,v_2,v_3\in V$ and $\alpha=B'(v_1,v_2)=cB(v_1,v_2)$, and $\beta=B'(v_1,v_3)=dB(v_1,v_3)$, where $B(v_1,v_2)B(v_1,v_3)\neq 0$. Then,

(1) $B(v_1,\beta cv_2-\alpha dv_3)=\beta cB(v_1,v_2)-\alpha dB(v_1,v_3)=\beta B'(v_1,v_2)-\alpha B'(v_1,v_3)=0.$

Thus,

(2) $B'(v_1,\beta cv_2-\alpha dv_3)=0.$

The last equality in (1) implies $B'(v_1,\beta v_2-\alpha v_3)=0$ and thus

(3) $B'(v_1,\beta dv_2-\alpha dv_3)=0.$

Now (2) and (3) imply

$$0=B'(v_1,(\beta c-\beta d)v_2)=\beta(c-d)B'(v_1,v_2)=\beta c(c-d)B(v_1,v_2)$$
$$=cd(c-d)B(v_1,v_2)B(v_1,v_3).$$

Therefore $cd(c-d)=0$. If $d=0$, then $B'(v_1,v_3)=0$ and if $B(v_1,v_2)=\lambda B(v_1,v_3)$, then $B(v_1,v_2-\lambda v_3)=0$ implies $B'(v_1,v_2-\lambda v_3)=0$ implies $B'(v_1,v_2)=\lambda B'(v_1,v_3)=0$. Thus $c=0$. Similarly $c=0$ implies $d=0$ so that we always have $c=d$.

Thus, for each $v_1\in V$ there is a c such that $B'(v_1,v)=cB(v_1,v)$ for all $v\in V$. Therefore, given $v_1'\in V$, there is a c' such that $B'(v_1',v)=c'B(v_1',v)$ for all $v\in V$. If $B(v_1',v)=0$ for all v, we can assume $c'=c$. Otherwise, choose v_2' so that $B(v_1',v_2')\neq 0$. By choosing w from v_2,v_2',v_2+v_2' we can ensure that $B(v_1,w)\neq 0$ and $B(v_1',w)\neq 0$. Applying our earlier argument to $\bar{B}(x,y)=B(y,x)$, we have $c=c'$. Thus, $B'=cB$.

Richard Beigel, student
Stanford University

Also solved by the proposer. Enzo R. Gentile (Argentina) solved the problem for V finite dimensional. Mark F. Kruelle pointed out the misstatement and deduced $B=cB'$ from ($B(x,y)=0$ if and only if $B'(x,y)=0$).

An Ancient Method September 1980

1103. For which positive integers does there exist a sequence of n consecutive integers for which the jth integer, $1\leqslant j\leqslant n$, has at least j divisors, none of which divides any other member of the sequence? [*Hal Forsey, San Francisco State University.*]

Solution: The result is true for all n. Given n, let p_1 be a prime larger than n, let p_2 be a prime larger than p_1, etc. Now let $q_1=p_1$, $q_2=p_2p_3$, $q_3=p_4p_5p_6$, and, in general, $q_k=p_{n_k+1}\cdots p_{n_{k+1}}$, where $n_k=(k-1)k/2$. By the Chinese Remainder Theorem, we can find an x such that $x+i\equiv 0 \pmod{q_i}$ for $i=1,\ldots,n$. The sequence $x+1,x+2,\ldots,x+n$ is a sequence of type desired; for $x+j$ is divisible by each of the j primes whose product is q_j, and since each p_i is larger than n, it is a factor of just one term in the sequence.

Richard A. Gibbs
Fort Lewis College

Also solved by Richard Beigel, L. Kuipers (Switzerland), Jerry Metzger, St. Olaf Problem Solving Group, J. L. Selfridge, and the proposer. One reader interpreted the "none of which" to mean "none of its prime divisors" rather than the intended "none of these j divisors" and deduced, of course, that n must be less than or equal to three.

1104. For n equally spaced points on the unit circle, consider the $\binom{n}{3}$ triangles formed by choosing three points at a time as vertices.

(a) Find S_n, the sum of the areas of all these triangles.

(b) If A_n denotes the average area of these triangles, determine limit A_n as $n \to \infty$. [*Nick Franceschine, Sebastopol, California.*]

Solution: Let $c = \exp(2\pi i/n)$. Then (c^r, c^s, c^t) with $0 \leqslant r < s < t \leqslant n-1$ are the vertices of the $\binom{n}{3}$ triangles. The area of such a triangle is

$$A(r,s,t) = \frac{1}{2}\mathrm{Im}\left[(c^t - c^r)(\bar{c}^s - \bar{c}^r)\right] = \frac{1}{2}\mathrm{Im}(c^{t-s} - c^{t-r} - c^{r-s}).$$

Hence,

$$4iA(r,s,t) = (c^{t-s} - c^{s-t}) + (c^{r-t} - c^{t-r}) + (c^{s-r} - c^{r-s}).$$

and

$$S_n = \sum A(r,s,t) = \sum_{t=2}^{n-1}\sum_{s=1}^{t-1}\sum_{r=0}^{s-1} A(r,s,t).$$

To compute S_n we use $c^n = 1$, $c\bar{c} = 1$, $\sum_{k=0}^{m} x^k = (1 - x^{m+1})/(1-x)$, and

$$\sum_{k=1}^{m} kx^k = x\left[1 - (m+1)x^m + mx^{m+1}\right]/(1-x)^2.$$

We find

$$\sum(c^{t-s} - c^{s-t}) = \sum(c^{s-r} - c^{r-s}) = \binom{n}{2}\frac{1+c}{1-c},$$

and $\sum(c^{r-t} - c^{t-r}) = n\dfrac{1+c}{1-c}$. Therefore,

$$S_n = \frac{n^2}{4}\frac{\mathrm{Im}\, c}{1 - \mathrm{Re}\, c} = \frac{n^2}{4}\cot\frac{\pi}{n}.$$

Finally,

$$\operatorname*{limit}_{n\to\infty} S_n \Big/ \binom{n}{3} = \frac{3}{2\pi}.$$

J. C. BINZ
University of Bern
Bern, Switzerland

Also solved by Alberto Facchini (Italy), Victor Hernandez (Spain), Mark Kruelle, L. Kuipers (Switzerland), John J. Martinez, Roger B. Nelsen, L. Van Hamme (Belgium), Doug Van Wieren, and the proposer.

Maximal Sums of Areas **September 1980**

1105.* For n points on the unit circle, consider the sum of the areas of the $\binom{n}{3}$ triangles formed by choosing three points at a time as vertices. Show that the sum is a maximum when the points are equally spaced. [*Peter Ørno, The Ohio State University.*]

Solution: [Revised by the editors.] Consider an n-gon inscribed in the unit circle. Let the sides of the polygon cut off successive arcs of lengths $\theta_1, \theta_2, \ldots, \theta_n$. Now the area of a triangle cutting off two arcs, of lengths α and β, is $\frac{1}{2}(\sin\alpha + \sin\beta - \sin(\alpha+\beta))$. Thus the sum of the areas of the $\binom{n}{3}$ triangles is

$$\sum_{j=1}^{[n/2]} \sum_{k=1}^{n} a_j \sin(\theta_k + \theta_{k+1} + \cdots + \theta_{k+j-1}), \tag{1}$$

where a_j depends on the number of triangles having a side which cuts off the arc of length $\theta_k + \theta_{k+1} + \cdots + \theta_{k+j-1}$. (This is independent of k.) Since the length of the chord cutting off an arc of length α is $2\sin(\alpha/2)$, $\Sigma_{k=1}^{n}\sin(\theta_k + \theta_{k+1} + \cdots + \theta_{k+j-1})$ is, for each j, one half of the perimeter of a nonsimple polygon whose arcs have lengths $2(\theta_k + \theta_{k+1} + \cdots + \theta_{k+j-1})$ and so goes around the inside of the circle $2j$ times.

For a regular n-gon, (1) becomes

$$\sum_{j=1}^{[n/2]} \sum_{k=1}^{n} a_j \sin j\theta, \text{ where } \theta = \frac{2\pi}{n}. \tag{2}$$

Since the regular n-gon has for each j the largest perimeter of any n-gon going around the circle $2j$ times [this can be proved by using Lagrange multipliers], each term of (2) is at least as large as the corresponding term of (1). This establishes the result.

John J. Martinez
St. Joseph's Prep
Philadelphia, Pa.

Permuting Decimal Expansions **September 1980**

1106. Let p: $N \to N$ be a permutation of the positive integers. Let $x = 0.a_1a_2\ldots$ be the decimal expansion (terminating, if possible) of x in $(0,1)$ and define $p^*(x) = 0.a_{p(1)}a_{p(2)}\ldots$. Thus, each permutation p induces a function p^*: $(0,1) \to (0,1)$. Characterize those permutations p for which p^* has at least one point of differentiability in $(0,1)$. [*Michael W. Ecker, Pennsylvania State University.*]

Solution: p^* has at least one point of differentiability if and only if p moves only a finite number of positive integers.

In fact if p moves a finite number of integers, i.e., for a fixed number n_0 $p(n) = n$ for all $n \geqslant n_0$, then $p^*(x) = x$ for all $x \in (0, 10^{-n_0})$ and therefore p^* is differentiable in $(0, 10^{-n_0})$.

Conversely, suppose p^* is differentiable at $x = 0.a_1a_2\ldots$. Define ϵ_n by $\epsilon_n = 10^{-n}$ if $0 \leqslant a_n \leqslant 4$ and $\epsilon_n = -10^{-n}$ if $5 \leqslant a_n \leqslant 9$. Then $\epsilon_n \to 0$ if $n \to \infty$ and since p^* is differentiable at x, the sequence $(p^*(x+\epsilon_n) - p^*(x))/\epsilon_n$ is convergent. The decimal expansion of $x + \epsilon_n$ is

$$0.a_1a_2\ldots a_{n-1}(a_n \pm 1)a_{n+1}\ldots(+ \text{ if } 0 \leqslant a_n \leqslant 4, \ - \text{ if } 5 \leqslant a_n \leqslant 9).$$

It follows that

$$(p^*(x+\epsilon_n) - p^*(x))/\epsilon_n = 10^{-p^{-1}(n)}/10^{-n} = 10^{n-p^{-1}(n)}.$$

Hence the sequence $10^{n-p^{-1}(n)}$, $n \in N$, is convergent. Therefore either limit $(n - p^{-1}(n))$ is an integer or limit $(n - p^{-1}(n)) = -\infty$.

In both cases of limit $(n - p^{-1}(n)) < 0$, there exists $n_1 \in N$ such that $n - p^{-1}(n) < 0$ for all $n \geqslant n_1$, from which $p^{-1}(n) > n$ for all $n \geqslant n_1$. This contradicts the bijectivity of p^{-1}. Similarly, if limit $(n - p^{-1}(n)) > 0$. Hence limit $(n - p^{-1}(n)) = 0$. Since $n - p^{-1}(n)$ is an integer for all $n \in N$, there exists $n_0 \in N$ such that $n - p^{-1}(n) = 0$ for all $n \geqslant n_0$. Therefore p^{-1} moves only a finite number of integers and the same holds for p.

Alberto Facchini
University of Padova
Padova, Italy

Also solved by Richard Beigel and the proposer.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Pierre J. Malraison, Jr., Editor
MDSI, Ann Arbor

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Rucker, Rudy, *Martin Gardner, impresario of mathematical games*, Science 81 2:6 (July/August 1981) 32-37, 102.

Gardner, on his impending retirement from his Mathematical Games column: "I have no desire to be in *Who's Who* or to go about the country making speeches. I just want to live quietly and anonymously in the mountains of western North Carolina and write the sort of books I wanted to write before I got sidetracked."

Simon, Herbert A., *Studying human intelligence by creating artificial intelligence*, American Scientist 69 (May-June 1981) 300-309.

Nobel-prize winner illustrates "how the techniques of computer simulation are being used today to gain a deeper understanding of human intelligence and human cognitive processes." The underlying hypothesis is that the brain operates by using the same basic operations on symbols as computers (read, write, store, compare and branch), and that such symbol-processing ability is the essence of intelligence.

Herman, Ros, *The mathematics of a rubber band*, New Scientist 89 (19 February 1981) 480-482.

A small proportion of crosslinks between polymer chains makes rubbers solid instead of liquid, and their presence can be modelled to satisfactorily account for the behavior of a rubber band. Very surprisingly, similar analysis for glass, which has more crosslinks, produces problems resembling those from the theory of elementary particles and its nonabelian gauge groups.

Sher, George, *What makes a lottery fair?*, Noûs 14 (1980) 203-216.

Investigates in an intuitive fashion what it means for a lottery to be fair, including the knowledge and psychological state of the contestants; then argues that it is morally preferable to use fair lotteries to allocate indivisible goods to which several people have equal claims.

Everhart, Laurence, *Sliding into the past: slide rules and calculators*, Antiques Journal 35 (1980) 30-33, 51.

"It is probable that the slide rule will disappear, and this may be time to consider collecting them." The article depicts and describes a dozen or so, and notes that the device was invented in 1859 by Amadée Mannheim, a French artillery officer. Will early pocket calculators be antiques some day, too?

Fossum, Timothy V. and Lewis, Gilbert N., *A mathematical model for trailer-truck jackknifing*, SIAM Review 23 (1981) 95-99.

Model uses vector calculus, results in a theorem, and proves satisfying.

Frauenthal, J.C., Mathematical Modeling in Epidemology, Springer-Verlag, 1980; vii + 118 pp.

This book's main purpose is not the study of epidemology but the illustration of the process of formulating and solving mathematical models, deterministic and stochastic. It arose from a course for senior mathematical sciences majors; it includes exercises but not modeling projects.

Burghes, D.N. and Borrie, M.S., Modelling with Differential Equations, Halsted, 1981; 172 pp, $29.95.

Stimulating assortment of applications, arranged by type of equation. Excellent material to supplement unit on differential equations in a calculus course, except for the ridiculous price (presumably caused by printing in Great Britain).

Abelson, Harold and diSessa, Andrea, Turtle Geometry: The Computer as a Medium for Exploring Mathematics, MIT Pr, 1981; xx + 477 pp, $20.

The authors claim that the book is "a computer-based introduction to geometry and advanced mathematics at the high school or college level;" but both the reading level and the mathematical sophistication expected of the reader mark it as a text for advanced undergraduates or mathematics teachers continuing their education. It is a *splendid* book, in which one is led to modeling, space-filling curves, vectors, deformations of curves and planes, spherical geometry, maze-solving algorithms, intrinsic curvature, and relativity!

Hamming, Richard W., Coding and Information Theory, Prentice-Hall, 1980; xii + 239 pp.

Excellent and entertaining survey by one of the founders of the field, employing a diversity of undergraduate mathematics, from the law of large numbers to Markov chains.

Barr, Avron and Feigenbaum, Edward A., eds., The Handbook of Artificial Intelligence, Vol. I, Kaufmann, 1981; xiv + 309 pp, $30.

First of 3 volumes of 200 articles about key concepts, important programming techniques, and achievements of AI. This volume offers overviews plus many articles under the broad headings of search, knowledge, and understanding natural and spoken languages.

Eves, Howard, Great Moments in Mathematics (Before 1650), Dolciani Mathematical Expositions No. 5, MAA, 1980; xiv + 270 pp, $22.

Pared-down versions of 20 lectures from an exciting course presenting a panoramic overview of mathematics in popular fashion. One only regrets the lack of the props and visual aids that accompany the oral presentations.

Davis, Philip, Jr. and Hersh, Reuben, The Mathematical Experience, Birkhauser, 1980; xxi + 440 pp, $24.

"What is the nature of mathematics? What is its meaning? What are its concerns? What is its methodology? How is it created and used?" In many short essays, liberally illustrated with figures and photographs, the authors provide a comprehensive survey of mathematics and its philosophy that will interest mathematicians and the cultured public alike. The title of their last essay, "True Facts about Imaginary Objects," sums up their rejection of Platonism in favor of conceptualism, a viewpoint attacked by Martin Gardner in his review, "Is mathematics for real?", *New York Review of Books* (13 August 1981).

Kingman, J.F.C., Mathematics of Genetic Diversity, SIAM, 1980; vii + 70 pp.

Lectures from a CBMS-NSF Regional Conference at Iowa State, offering models of selection and mutation.

NEWS & LETTERS

DAN J. EUSTICE

Dan J. Eustice, Problems Editor of *Mathematics Magazine*, died suddenly of a heart attack on August 5, 1981. Dan was Associate Professor of Mathematics at Ohio State University, and was the author of a number of papers in various fields of analysis. He was 50 years old, and left a wife and three children.

Dan (not "Daniel"--he claimed to be named for an entire tribe) appeared to be in splendid health. He exercised regularly and vigorously. That morning he had ridden his bicycle several miles from home to the Ohio State campus, and he had scheduled a handball game for late that afternoon. He was in good spirits, being especially pleased with the number and quality of the incoming freshmen who would start in the Honors Calculus sequence in the Autumn Quarter.

Dan's talents and interests made him well suited for his position as Problems Editor. He taught not only motivated honors students, but also many students with poor attitudes in "Approaches to Mathematics" courses; his use of challenging problems, games, and puzzles worked effectively, whatever his audience. He coached Ohio State's Putnam Team and served as grader for the Putnam Examination. Dan's session on Rubik's Cube at a High School Visitation day at Ohio State was jammed with students, cubes in hand, whose interest verged on the fanatic.

Dan tended to be quiet outside the classroom. He was well liked and gentlemanly. His sense of humor was quite in keeping with his aptitude for games. He enjoyed jokes and double-crostics, and there was never a pun too atrocious for him to repeat.

His many friends in the mathematical community mourn his passing.

Editor's note: We are grateful to Frank Carroll, a close friend and colleague of Dan at Ohio State for providing this warm remembrance.

FORGOTTEN GEOMETRY

The geometry blunder (reported in "Mathematical Sunshine," this *Magazine*, May 1981, p. 152) was an illustration of the low depths to which knowledge of Euclidean geometry (especially three dimensional) has sunk. It used to be a well-known fact that the plane angles of the dihedral angles of the regular tetrahedron and regular octahedron are supplementary. This implies that when a triangular face of the triangular pyramid (regular tetrahedron) is made congruent with a triangular face of the square pyramid (half a regular octahedron), the faces adjacent to the common face but not in the same original solid are coplanar. Hence the resulting solid has five faces and is in fact a triangular prism.

If one does not know of the supplementary relationship, it is easy to see it. The cosine of the plane angle of the dihedral angle of a regular tetrahedron is easily seen to be 1/3. It can also be seen that the cosine of the plane angle θ of the dihedral angle between the square base and one of the triangular faces of the square pyramid is $1/\sqrt{3}$. Since the angle between two triangular faces of the square pyramid is 2θ, the cosine of this angle is -1/3. Hence the supplementary relationship.

John P. Hoyt
1115 Marietta Ave., 32
Lancaster, PA 17603

SOLUTIONS TO 1981 U.S.A. AND CANADIAN MATHEMATICAL OLYMPIADS

The following solutions to the 1981 U.S.A. and Canadian Mathematical Olympiad problems were prepared by Loren C. Larson, St. Olaf College. We will publish solutions to the 1981 International Olympiad problems in January. A pamphlet with solutions and information on scores, prepared by Samuel Greitzer, is available for $.50 from Dr. Walter E. Mientka, MAA Committee on High School Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, NB 68588.

U.S.A. OLYMPIAD

1. The measure of a given angle is $180°/n$ where n is a positive integer not divisible by 3. Prove that the angle can be trisected by Euclidean means (straightedge and compass).

Sol. Since n and 3 are relatively prime, there are integers s and t such that $sn + 3t = 1$. Multiplying by $60/n$ yields $60/n = 60s + (180/n)t$. We are given an angle of measure $180/n$ and we can construct an angle of measure 60, and therefore, the last equation describes how to construct $60/n$.

2. Every pair of communities in a county are linked directly by exactly one mode of transportation: bus, train, or airplane. All three modes of transportation are used in the county with no community being serviced by all three modes and no three communities being linked pairwise by the same mode. Determine the maximum number of communities in the county.

Sol. Four communities, A, B, C, D can be connected according to the required stipulations in the following way: bus, AB, BC, CD, DA; train, AC; airplane, BD. An easy argument shows that no community can have a single mode of transportation leading to each of three different communities. Therefore, there can be at most five communities. Suppose, then, that A, B, C, D, E are five communities satisfying the stated conditions. Suppose that A is connected to B and C by mode S and to D and E by mode T. Two mode S routes leave C; we may assume that one of them leads to D. Communities D and E cannot be connected by mode T, therefore they must be linked by mode S. It follows that C and E must be connected by mode T, and this implies that B and C are linked by mode T. Continuing, we see that only two modes of transportation are used in the county, contrary to one of the requirements. Therefore, five communities cannot be linked in the desired manner, and consequently, there are at most four communities in the county.

3. If A, B and C are the measures of the angles of a triangle, prove that $-2 \le \sin 3A + \sin 3B + \sin 3C \le 3\sqrt{3}/2$ and determine when equality holds.

Sol. We are given $A + B + C = 180$ and we may assume $A \geq B > C \geq 0$. It follows that $0 < C \le 60$ and therefore $\sin 3A + \sin 3B + \sin 3C \geq \sin 3A + \sin 3B > -2$. For the other inequality, we need only be concerned with the case in which $120 < A < 180$. Then $0 < C \le B < 60$ and it follows from the concavity of $\sin x$ on $[0,180]$ that $\sin 3B + \sin 3C \le 2 \sin \frac{3B+3C}{2} = -2 \cos \frac{3}{2}A$. Hence $\sin 3A + \sin 3B + \sin 3C \le \sin 3A - 2 \cos \frac{3}{2}A = 2 \cos \frac{3}{2}A\,[\sin \frac{3}{2}A - 1]$. Now, let x be defined by $A = 140 + 2x/3$, $-30 < x < 60$. Then, the last expression is $2\cos(210 + x)[\sin(210 + x) - 1]$, which a lengthy calculation shows is equal to

$$\frac{1}{2} + \left(\frac{3\sqrt{3}-1}{2}\right) \cos^2 x - \left(\frac{\sqrt{3}-1}{2}\right)(\cos x - 1)^2 - \frac{1}{2}(\sin x - \cos x + 1)^2.$$

In this form it is easy to see that the maximum occurs when $x = 0$ and is equal to $\frac{3\sqrt{3}}{2}$. Thus

$\sin 3A + \sin 3B + \sin 3C \le \frac{3\sqrt{3}}{2}$, with equality when $A = 140$, $B = C = 20$.

4. The sum of the measures of all the face angles of a given convex polyhedral angle is equal to the sum of the measures of all its dihedral angles. Prove that the polyhedral angle is a trihedral angle. Note: A convex polyhedral angle may be formed by drawing rays from an exterior point to all points of a convex polygon.

Sol. It is not difficult to show that the sum of the measures of the face angles of any convex polyhedral angle is less than 360, and the sum of the measures of the dihedral angles is larger than the sum of the interior angles of the polygon. Let n denote the number of sides in the polygon. From above, the given information implies that $(n - 2)180 < 360$, and this can happen only if $n = 3$.

5. If x is a positive real number and n is a positive integer, prove that

$$[nx] \geq \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \dots + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t. For example, $[\pi] = 3$ and $[\sqrt{2}] = 1$.

Sol. The problem is easily reduced to the case in which $0 < x < 1$.

For each positive integer m, let P_m be the statement that the desired inequality holds (for all n) for all values of x which have the form s/m, where s is an integer, $0 < s < m$. We will use induction to prove that P_m is true for all m.

An easy induction on n shows that P_2 is true. Suppose that P_k is true for $k < m$, and let $x = s/m$, where s is a positive integer less than m. If s and m are not relatively prime, the induction assumption shows that the desired inequality holds for all n. Therefore, assume that s and m are relatively prime.

Let $S = \{a/b\colon a$ and b are integers, $0 \le a < b < m$, $a/b < s/m\}$. S is a finite set; let c/d be the largest element in S, c and d integers, $0 \le c < d < m$. We claim that $[kc/d] = [ks/m]$ for $k = 1,2,\ldots,m-1$. Clearly, $[kc/d] \le [ks/m]$, so suppose that for some k, $[kc/d] < [ks/m]$. Then $c/d < ([kc/d] + 1)/k \le [ks/m]/k < s/m$ (the last inequality is strict because ks/m is not an integer). These inequalities imply that $[ks/m]/k$ is an element of S larger than c/d, contrary to our choice of c/d.

Suppose $n < m$. Then, using the results of the previous paragraph,

$$\sum_{k=1}^{n} \frac{[ks/m]}{k} = \sum_{k=1}^{n} \frac{[kc/d]}{k} \le [nc/d] \text{ (by the induction assumption)} \le [ns/m].$$

Suppose $n = m$. Then,

$$\sum_{k=1}^{n} \frac{[ks/m]}{k} \le \sum_{k=1}^{n} \frac{ks/m}{k} = s = [ns/m].$$

Suppose $n > m$. Write $n = qm + r$, q and r integers, $0 < r \le m$. Then for

$$x = s/m, \quad \sum_{k=1}^{n} \frac{[kx]}{k} = \sum_{k=1}^{r} \frac{[kx]}{k} + \sum_{i=0}^{q-1} \sum_{j=1}^{m} \frac{[(r+im+j)x]}{r+im+j}$$
$$\le [rx] + \sum_{i=0}^{q-1} \sum_{j=1}^{m} \frac{(r+im+j)x}{r+im+j}$$
$$= [rx] + mqx = [nx].$$

Thus, by induction, the result holds for all rational values of x.

Suppose that x is irrational, $0 < x < 1$. Fix n. Pick a rational number r, $0 < r < 1$, such that $kx < kr < [kx] + 1$ for $k = 1,2,\ldots,n$. Then $\sum_{k=1}^{n} \frac{[kx]}{k} = \sum_{k=1}^{n} \frac{[kr]}{k} \le [nr]$ (by preceding work) $= [nx]$ (by choice of r). This completes the proof.

CANADIAN OLYMPIAD

1. For any real number t denote by $[t]$ the greatest integer which is less than or equal to t. For example, $[8] = 8$, $[\pi] = 3$, and $[-5/2] = -3$. Show that the equation $[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345$ has no real solution.

Sol. Suppose x is a solution, and write x in the form $x = N + a/2 + b/4 + c/8 + d/16 + e/32 + f$, where N is an integer, a, b, c, d, e are each either 0 or 1, and $0 \le f < 1/32$. Our equation implies that $63N + 31a + 15b + 7c + 3d + e = 12345$. It follows that $63N \le 12345 < 64N$, and therefore $N = 195$ and $31a + 15b + 7c + 3d + e = 60$. It follows that $a = 1$ and $15b + 7c + 3d + e = 29$. But this is impossible, since $15b + 7c + 3d + e \le 26$.

2. Given a circle of radius r and a tangent line ℓ to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to ℓ with Q on ℓ. Determine the maximum of the area of triangle PQR.

Sol. Let $\theta = \angle RPQ$. Then Area $\Delta PQR = \frac{1}{2}|PR|^2 \cos\theta \sin\theta$. But $|PR|^2 = 2r^2 - 2r^2\cos 2\theta = 4r^2\sin^2\theta$, so that Area $\Delta PQR = 2r^2\sin^3\theta\cos\theta$. Let $\sin^2\theta = 3/4 + x$. Then $\sin^6\theta\cos^2\theta = (3/4 + x)^3 - (3/4 + x)^4 = (3/4)^3 - (3/4)^4 - x^2[1/8 + (1 + x)^2]$. In this form we see that the maximum value of $\sin^3\theta\cos\theta$ is $[(3/4)^3 - (3/4)^4]^{1/2} = 3\sqrt{3}/16$, so that the maximum area is $3\sqrt{3}\, r^2/8$.

3. Given a finite collection of lines in a plane P, show that it is possible to draw an arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, etc.) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

Sol. Let R be an arbitrary positive number. Suppose there are m lines. For $k = 1,2,\ldots,m+1$, let S_k denote the set of circles of radius R having centers at the coordinates $(5nR, 5kR)$, for $n = 1,2,\ldots$. A given line in the collection will intersect only a finite number of circles in S_k, unless the line is parallel to, and within a distance R of, their line of centers ($y = 5kR$). Since there are $m + 1$ sets S_k and only m lines in the collection, there is a k such that no line intersects all the circles in S_k. Since each line intersects only a finite number of circles in S_k, some circle in S_k does not meet any of the lines in the collection.

Let n be a positive integer. There exist unique nonnegative integers m and k, $0 \leq k < 2^m$, such that $n = 2^m + k$. Let L_n be the line through the origin whose angle (in radians) with the positive x axis is $(2k + 1)/2^{m+1}$. Clearly, every circle in P will meet one of these lines.

4. $P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P(Q(x)) \equiv Q(P(x))$ for all real numbers x. If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P(P(x)) = Q(Q(x))$ also has no real solution.

Sol. Suppose that $P(x) > Q(x)$ for all x. Then, for all x, $P(P(x)) > Q(P(x)) = P(Q(x)) > Q(Q(x))$. Similarly, $P(x) < Q(x)$ for all x implies that $P(P(x)) < Q(Q(x))$ for all x. The result follows, since if $P(x) = Q(x)$ has no real solutions, either $P(x) > Q(x)$ for all x, or $P(x) < Q(x)$ for all x.

5. Eleven theatrical groups participated in a festival. Each day some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last?

Sol. Let $A = (a_{ij})$ be a matrix defined by $a_{ij} = 1$ or 0 depending upon whether group i performs, or does not perform, respectively, on the jth day.

We shall make repeated use of the dual nature of the problem, namely, that for any collection of groups, each group has seen each of the others (of the collection) perform if and only if each group has performed for each of the other groups. Thus, for example, if a matrix A defines a satisfactory schedule, so will $\overline{A} = (\overline{a}_{ij})$ defined by $\overline{a}_{ij} = 0$ or 1 according as $a_{ij} = 1$ or 0, respectively.

Consider the case of scheduling three groups. By the dual nature discussed above, we may assume that exactly two groups perform on the first day. It will take two additional days for these two groups to hear each other, and therefore at least three days are required. It is easy to check that three days is sufficient.

In case of four groups, we may assume that either two or three groups perform on the first day. In either case, three additional days are required.

We will prove that at least six days are required to handle eleven groups. That six are sufficient is seen in the following matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

To see that six is necessary, observe that we may assume that six or more perform on the first day. If more than six perform on the first day, then we may assume that four or more of these perform on the second day, and from above, four additional days are required, making a total of at least six days.

Suppose then that exactly six perform on the first day. One can argue that, up to row permutations, these six can be scheduled in five days in only one way, as shown in the first six rows of the matrix above. In addition, we may assume that exactly two groups will not perform on either the first or second day (last two rows of the above matrix). Then, one can argue that these two groups, together with the above six, cannot be satisfactorily scheduled in five days.

ACKNOWLEDGEMENTS

In addition to our associate editors, the following have assisted the *Magazine* by refereeing papers during the past year. We appreciate their special efforts.

Alexanderson, Gerald, *University of Santa Clara.*
Anderson, Sabra, *University of Minnesota, Duluth.*
Anton, Howard, *Drexel University.*
Arnold, Leslie K., *Daniel H. Wagner, Associates.*
Assmus, Edward, *Lehigh University.*
Banchoff, Thomas F., *Brown University.*
Bankoff, Leon, *Los Angeles, CA.*
Baxley, John, *Wake Forest University.*
Beals, Richard W., *Yale University.*
Berggren, John L., *Simon Fraser Univ.*
Boas, Mary L., *DePaul University.*
Boas, Ralph P., *Northwestern Univ.*
Boyd, David, *University of British Columbia.*
Brown, Walter, *Moravian College.*
Brubaker, Marvin, *Moravian College.*
Burckel, Robert, *Kansas State Univ.*
Burton, David M., *University of New Hampshire.*
Cebulka, Kathy Dickens, *Annandale, NJ.*
Chakerian, Don, *University of California, Davis.*
Cochran, Jane, *Moravian College.*
Coen, Robert, *York College (CUNY).*
Coleman, Roger, *Ursinus College.*
Connelly, Robert, *Cornell University.*
Coxeter, H.S.M., *University of Toronto.*
Crowe, Donald, *University of Wisconsin.*
Davison, Thomas M.K., *McMaster Univ.*
Dershem, Herbert L., *Hope College.*
Diamond, George, *Moravian College.*
Dixon, Crist, *York College (CUNY).*
Donaldson, James A., *Howard University.*
Dow, Paul, *Moravian College.*
Dunham, Douglas, *University of Minnesota, Duluth.*
Eggan, Lawrence C., *Illinois State University.*
Eisenberg, Bennett, *Lehigh University.*
Emerson, John D., *Middlebury College.*
England, James, *Swarthmore College.*
Epp, Susanna, *DePaul University.*
Fisher, J. Chris, *University of Regina.*
Flesner, David E., *Gettysburg College.*
Foster, Lorraine L., *California State University at Northridge.*
Galovich, Steven, *Carleton College.*
Getchell, Charles L., *Lycoming College.*
Gill, Tepper, *Howard University.*
Gilmer, Robert, *Florida State University.*
Goldman, Jay, *University of Minnesota.*
Gould, Henry, *West Virginia University.*
Greenstein, Barry, *Univ. of Illinois.*
Grosshans, Frank D., *West Chester State College.*
Growney, JoAnne, *Bloomsburg State College.*
Grünbaum, Branko, *Univ. of Washington.*
Guy, Richard, *University of Calgary.*
Hailperin, Ruth, *Moravian College.*
Hailperin, Theodore, *Lehigh University.*
Halmos, Paul R., *Indiana University at Bloomington.*
Henricksen, Melvin, *Harvey Mudd College.*
Heuer, Gerald A., *Concordia College.*
Honsberger, Ross A., *University of Waterloo.*
Hutchinson, Joan, *Smith College.*
Infante, Ronald P., *Seton Hall Univ.*
Kelly, Leroy M., *Michigan State Univ.*
Kline, Morris, *New York University, Courant Institute.*
Klotz, Eugene, *Swarthmore College.*
Konhauser, Joseph D.E., *Macalester College.*
Koppelman, Elaine, *Goucher College.*
Kullman, David E., *Miami University.*
Laatsch, Richard, *Miami University.*
Landau, Martin D., *Lafayette College.*
LaRocque, Gerald R., *Arthur D. Little.*
Lee, Jae, *Moravian College.*
Leinbach, Carl, *Gettysburg College.*
Lewand, Robert, *Goucher College.*
Lewis, Jesse C., *Jackson State Univ.*
Logothetti, David, *University of Santa Clara.*
Madachy, Joseph S., *Kettering, OH.*
Maki, Daniel, *Indiana University at Bloomington.*
Martens, Susan, *York College (CUNY).*
Mattson, H.F., Jr., *Syracuse University.*
McAllister, Marialuisa, *Moravian College.*
Mendelsohn, Eric, *University of Toronto.*
Miller, Kenneth S., *Riverside Research Institute.*
Mills, George H., *St. Olaf College.*
Moody, Robert V., *University of Saskatchewan.*
Moore, David S., *Purdue University.*
Moore, L.C., Jr., *Duke University.*
Mugler, Dale H., *Univ. of Santa Clara.*
Murphy, Michael G., *University of Houston, Downtown College.*
Nijenhuis, Albert, *University of Pennsylvania.*
Niven, Ivan, *University of Oregon.*
Notz, William I., *Purdue University.*
Ott, Johanna, *Moravian College.*
Paul, J., *University of Cincinnati.*
Pierce, Keith, *University of Minnesota, Duluth.*
Porter, Gerald J., *U. of Pennsylvania.*

AUTHORS

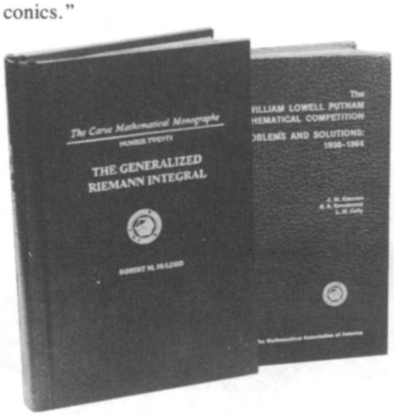
FOUR NEW BOOKS FROM THE MAA...

MAA Studies in Mathematics #20

STUDIES IN ALGEBRAIC GEOMETRY
A. Seidenberg, editor
143 + ix pages hardbound.
List: $16.00 MAA Member: $12.00

STUDIES IN ALGEBRAIC GEOMETRY, a new volume in the MAA Studies in Mathematics Series, contains expository articles which make some of the most interesting results in algebraic geometry accessible to students, as well as teachers and mathematicians who are nonspecialists or scientists working in other fields.

The contents include chapters on "initial results in the theory of linear algebraic groups," "the connectedness theorem and the concept of multiplicity," "space curves as ideal-theoretic complete intersections," and "Chasle's enumerative theory of conics."



Carus Mathematical Monographs #20

THE GENERALIZED RIEMANN INTEGRAL
by Robert M. McLeod
275 + xiii pages hardbound.
List: $18.00 MAA Member: $13.50

THE GENERALIZED RIEMANN INTEGRAL is the first book-length presentation of an exciting new development in integration theory. A new integral, a generalization of the familiar Riemann integral, has been discovered which has all the power and range of the Lebesque integral but which can be readily understood by anyone who has studied calculus through the sophomore level.

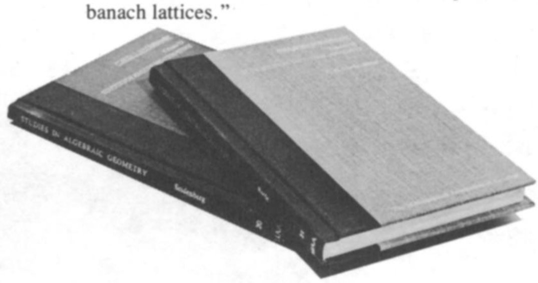
Because of the clarity and organization of the exposition and the inclusion of exercises designed to actively engage the reader in the material, it is eminently suitable for use as a textbook at the advanced undergraduate or beginning graduate level.

MAA Studies in Mathematics #21

STUDIES IN FUNCTIONAL ANALYSIS
Robert Bartle, editor
227 + xi pages hardbound.
List: $19.00 MAA Member: $14.00

STUDIES IN FUNCTIONAL ANALYSIS, a new addition to the MAA's series STUDIES IN MATHEMATICS, presents five expository articles on recent developments in functional analysis. Although these articles cannot provide a full account of the current state of the field, they do give a reader a tantilizing taste of some of the latest ideas and applications of this fascinating area of mathematics.

The contents include chapters on "numerical ranges," "projection operators in approximation theory," "complementably universal separable banach spaces; an application of counter examples to the approximation problem," "integral representation for elements of convex sets," and "aspects of banach lattices."



A New Problem Book from the MAA...

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION: PROBLEMS AND SOLUTIONS: 1938-1964
A. M. Gleason, R. E. Greenwood, and L. M. Kelly
652 + xi pages hardbound.
List: $35.00 MAA Member: $26.00

Problemists the world over, including all past and future Putnam Competitors, will revel in "mastering the difficulties" posed by this collection of problems from the first 25 William Lowell Putnam Competitions. Solutions to all 347 problems are given. In some cases multiple solutions are included, some which contestants could reasonably be expected to find under examination conditions and others which are more elegant or utilize more sophisticated techniques. Valuable references and comments on many of the problems are presented.

ORDER FROM:
The Mathematical Association of America
1529 Eighteenth Street, N.W.
Washington, D.C. 20036

U.S. POSTAL SERVICE

STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION

(Required by 39 U.S.C. 3685)

1. TITLE OF PUBLICATION: MATHEMATICS MAGAZINE
A. PUBLICATION NO: M A M C A 8
2. DATE OF FILING: Sept. 14, 1981
3. FREQUENCY OF ISSUE: every two months except July and August
A. NO. OF ISSUES PUBLISHED ANNUALLY: Five
B. ANNUAL SUBSCRIPTION PRICE: Library $18.00; Indiv. Member $8.00
4. COMPLETE MAILING ADDRESS OF KNOWN OFFICE OF PUBLICATION *(Street, City, County, State and ZIP Code) (Not printers)*: 1529 18th Street, NW, Washington, DC 20036
5. COMPLETE MAILING ADDRESS OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS *(Not printers)*: 1529 18th Street, NW, Washington, DC 20036
6. FULL NAMES AND COMPLETE MAILING ADDRESS OF PUBLISHER, EDITOR, AND MANAGING EDITOR *(This item MUST NOT be blank)*

PUBLISHER *(Name and Complete Mailing Address)*: The Mathematical Association of America, 1529 18th St., NW, Washington, DC 20036

EDITOR *(Name and Complete Mailing Address)*: Doris Schattschneider, Moravian College, Bethlehem, PA 18018

MANAGING EDITOR *(Name and Complete Mailing Address)*: Dr. Raoul Hailpern, SUNY at Buffalo, Buffalo, NY 14214

7. OWNER *(If owned by a corporation, its name and address must be stated and also immediately thereunder the names and addresses of stockholders owning or holding 1 percent or more of total amount of stock. If not owned by a corporation, the names and addresses of the individual owners must be given. If owned by a partnership or other unincorporated firm, its name and address, as well as that of each individual must be given. If the publication is published by a nonprofit organization, its name and address must be stated.) (Item must be completed)*

FULL NAME	COMPLETE MAILING ADDRESS
The Mathematical Association of America, Inc.	1529 18th St., NW Washington, DC 20036

8. KNOWN BONDHOLDERS, MORTGAGEES, AND OTHER SECURITY HOLDERS OWNING OR HOLDING 1 PERCENT OR MORE OF TOTAL AMOUNT OF BONDS, MORTGAGES OR OTHER SECURITIES *(If there are none, so state)*

FULL NAME	COMPLETE MAILING ADDRESS
None	

9. FOR COMPLETION BY NONPROFIT ORGANIZATIONS AUTHORIZED TO MAIL AT SPECIAL RATES *(Section 411.3, DMM only)*
The purpose, function, and nonprofit status of this organization and the exempt status for Federal income tax purposes *(Check one)*

[X] (1) HAS NOT CHANGED DURING PRECEDING 12 MONTHS
[ ] (2) HAS CHANGED DURING PRECEDING 12 MONTHS
(If changed, publisher must submit explanation of change with this statement.)

10. EXTENT AND NATURE OF CIRCULATION	AVERAGE NO. COPIES EACH ISSUE DURING PRECEDING 12 MONTHS	ACTUAL NO. COPIES OF SINGLE ISSUE PUBLISHED NEAREST TO FILING DATE
A. TOTAL NO. COPIES *(Net Press Run)*	10,100	10,000
B. PAID CIRCULATION 1. SALES THROUGH DEALERS AND CARRIERS, STREET VENDORS AND COUNTER SALES	none	none
2. MAIL SUBSCRIPTION	8,361	8,183
C. TOTAL PAID CIRCULATION *(Sum of 10B1 and 10B2)*	8,361	8,183
D. FREE DISTRIBUTION BY MAIL, CARRIER OR OTHER MEANS SAMPLES, COMPLIMENTARY, AND OTHER FREE COPIES	40	40
E. TOTAL DISTRIBUTION *(Sum of C and D)*	8,401	8,223
F. COPIES NOT DISTRIBUTED 1. OFFICE USE, LEFT OVER, UNACCOUNTED, SPOILED AFTER PRINTING	1,699	1,777
2. RETURN FROM NEWS AGENTS	none	none
G. TOTAL *(Sum of E, F1 and 2 - should equal net press run shown in A)*	10,100	10,000

11. I certify that the statements made by me above are correct and complete
SIGNATURE AND TITLE OF EDITOR, PUBLISHER, BUSINESS MANAGER, OR OWNER: Alfred B. Willcox

PS Form June 1980 3526 (Page 1) *(See instruction on reverse)*

MAA STUDIES IN MATHEMATICS

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, N.W.
Washington, DC 20036

MATHEMATICS MAGAZINE VOL. 54, NO. 5, NOVEMBER 1981